SOME THEOREMS ON BINARY TOPOLOGICAL ALGEBRAS

By
DESMOND ALEXANDER ROBBIE

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
1970

ACKNOWLEDGMENTS

The author wishes to record his deep feelings of respect, gratitude and admiration towards the chairman of his supervisory committee, Professor Alexander Doniphan Wallace, the founder of modern topological algebra. Not only has Professor Wallace carried out the duties associated with his position to the maximum extent, but he has also been a mentor and friend, giving readily of his time and energy, even when inconvenient personally to himself. His continual inspiration has been the driving force behind the work in this dissertation.

The author also wishes to record his special gratitude for the help, friendship, and extra efforts of the co-chairman of his supervisory committee, Professor K. N. Sigmon, who was particularly encouraging in the early stages and always ready to discuss problems.

Special thanks are due also to Professor D. P. Stadtlander and Professor A. R. Bednarek for their general encouragement, and their freely given help whenever called upon.

The author is, as well, grateful for the general helpfulness of all members of his supervisory committee, and, indeed to all faculty members and fellow graduate students for the many discussions which helped to stimulate the production of the work in this dissertation.

Finally, the author is indeed most grateful for the special efforts of his typist, Jeanne Ojeda, who performed a first rate job in a short time.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in Partial Fulfillment
of the Requirements for the Degree of Doctor of Philosophy

SOME THEOREMS ON BINARY
TOPOLOGICAL ALGEBRAS

By

Desmond Alexander Robbie

December, 1970

Chairman: A. D. Wallace
Co-Chairman: K. N. Sigmon
Major Department: Mathematics

Basically, this dissertation is concerned with compact Hausdorff spaces on which there are defined one, or more, binary operations, each of which, considered as a function from the usual topological product of the space with itself into the space, is continuous. For simplicity in the text this is abbreviated to BAT (Binary Topological Algebra (singular)), or BATS (plural). Some of the work is for nonassociative operations, and, in general, although free use is made of many classical results of A. D. Wallace, and others, from topological semigroup (mob) theory, the general aim of the work is to expand, and explore, rather than to restrict, and specialise. Nonetheless, some highly specialised results are included, because they help to point up the need for new development.

Chapter I explains in more detail what has been outlined above.

Chapter II, the largest chapter, seeks to relate work from different BATS under the general notion of a compact universal BAT. To this end many special results in the areas of topological semirings, nearrings, and other closely related structures are given, and an attempt is made to knit the various ideas together into the general framework. Incidentally, some special results are obtained from the general viewpoint, including some concerning certain function spaces related to a compact topological group.

Chapter III contains answers to some specific problems of A. D. Wallace, and deals mainly with periodicity and recurrence.

Chapter IV concerns nonassociative algebra, and the results obtained there are, perhaps, the most interesting of all. There, connected BATS on ordered spaces with the order topology are studied. Briefly, the problem is to characterise such BATS as semigroups when they satisfy certain algebraic conditions such as varying degrees of power associativity. In particular, certain such BATS are shown to be iseomorphic to [0, 1/2] under ordinary multiplication. Also, a conjecture of P. S. Mostert is shown to be false, by means of a counter example.

CHAPTER I

INTRODUCTION

1. An Adumbration of What Is Meant by "Binary Topological Algebras (BATS)"

If we have a triple (T_1, T_2, T_3) of Hausdorff topological spaces, and also a $\begin{matrix}\text{(function)}\\ \text{(operation)}\end{matrix}$ $F\colon\ T_1 \times T_2 \to T_3$, which is continuous when $T_1 \times T_2$ is given the product topology induced by T_1 and T_2 in the usual way, then, we denote $F((t_1,t_2))$ by t_1t_2 (juxtaposition). It follows that for each open set $U(t_1t_2)$ in T_3 there exist open sets $U(t_1)$ and $U(t_2)$ in T_1 and T_2 respectively, such that

$$U(t_1)\ U(t_2) \subseteq U(t_1t_2)$$

(where $U(t_1)\ U(t_2)$ denotes the set of all products xy with $x \in U(t_1)$ and $y \in U(t_2)$).

Now it can happen that two or more of the three spaces are the same, for example,

$$T_1 \times T_1 \to T_1\ .$$

Also, it may happen that a space T_1 may participate in two such situations,

$$\text{A.}\ \begin{cases} F_1\colon & T_1 \times T_1 \to T_1 \\ F_2\colon & T_1 \times T_2 \to T_2 \qquad \text{(here } T_3 = T_2\text{),} \end{cases}$$

and we may write $F_1\ ((t_1,t_2))$ as $t_1 \circ t_2$. Then it may also happen that certain associativity conditions are present, for example,

$$t_1\,(t_1'\,t_1'') = (t_1 t_1')t_1'' \qquad \text{and/or} \tag{1}$$

$$t_1 \circ (t_1' \circ t_2) = (t_1 t_1') \circ t_2\ . \tag{2}$$

If we have situation (1), then we say that T_1 is a <u>topological semigroup</u> (or <u>mob</u>). If we have situation (2) and (1), we say that the <u>mob</u> T_1 <u>acts</u> on the space T_2 on the left. The system A is then called an <u>action</u>. (See Stadtlander [28], Day and Wallace [6], and Bednarek and Wallace [1].) However, we note that (2) is still of interest <u>without</u> (1).

We may also obtain situations where

$$\text{B.}\quad \begin{cases} F_2:\ T_2 \times T_2 \to T_2 \\ F_3:\ T_3 \times T_3 \to T_3 \\ F\ :\ T_1 \times T_2 \to T_3\ , \end{cases}$$

and then it can happen that (written shortly)

$$t_1\,(\underset{\substack{\uparrow \\ (\text{in } T_2)}}{t_2\,t_2'}) = (t_1\,t_2)\underset{\substack{\uparrow \\ (\text{in } T_3)}}{\ } (t_1\,t_2')\ . \tag{3}$$

Moreover, we can get more than one F_2 or F_3, and in such cases we usually write one of the operations on (a,b) as ab, and the other as a + b (but we do not always want the property a + b = b + a). When $T_1 = T_2 = T_3$, and we have two operations, then (3) might appear as

$$a\ (b + c) = ab + ac \qquad \text{(as in semirings).}$$

Here, juxtaposition stands for F in B and + stands for both F_2 and F_3. We can also do this even if $T_2 \neq T_3$ and/or $T_1 \neq T_2$ (making no distinction between the addition symbols when confusion is not likely).

Progress has also been made in cases where

$$F_1:\ T_1 \times T_1 \to T_1\ , \quad \text{and}$$

(4)
$$\begin{cases} t_1 + t_1' = t_1' + t_1 & \text{and} \\ t_1 + t_1 = t_1 & \text{are both satisfied.} \end{cases}$$

F_1 is then called a <u>mean</u> on T_1. Or,

(5)
$$(t_1\ t_1')\ (t_1''\ t_1''') = (t_1\ t_1'')\ (t_1'\ t_1''')\ ,$$

when it is said that F_1 is medial on T_1 (see Sigmon [27]).

Finally, we can have combinations of the situations in (1), (2), (3), (4) and (5), and, moreover, have feedback maps from T_3 into T_1, T_2, or both (see this dissertation, Chapter III); or, maps from T_1, T_2, or both, into T_3 (i.e., Wallace [34]). All of these situations arise and are to some extent fruitful.

2. General Objectives of This Dissertation

Included in the general concept of a BAT, are the concepts of topological semigroups with continuously acting operators, and, more particularly, groups with operators and rings with operators. The space of operators may or may not have some internal structure. For example, included in the notion of a topological semigroup with continuously acting operators is the concept of a semiring, and that of a semimodule over a semiring. Also covered in the general framework is the notion of a near ring, although, perhaps this is better thought of as a group with operators.

One object of this dissertation is to examine the ways that the different types of structures referred to above relate to one another, and to generalize as far as possible to produce theorems which may be thought of as BAT theorems from various theorems which appear only in certain types of BATS at present. Much of the work is for specially restricted topologies such as compact, or locally compact, locally connected, connected or metric, but maximum generality is aimed at.

A further object might be stated as being an attempt to delineate the area of BATS from that of purely algebraic considerations, to focus attention on those theorems that are more intimately related to the topological effect on the algebra.

In the course of these investigations solutions have been found to certain special problems of A. D. Wallace, and they are given here to help illuminate the general area.

3. General References

For general reference on those parts in the dissertation which involve some algebraic semigroup theory the reader is referred to Clifford and Preston [5]. For topological semigroups the reader is referred to the excellent expository book by Paalman- de Miranda [20], or, the treatise on compact connected semigroups with identity by Hofmann and Mostert [12].

CHAPTER II

COMPACT SEMIRINGS, NEARRINGS, AND COMPACT SEMIMODULES OVER COMPACT SEMIRINGS

1. Preliminary Definitions

From now on ab will stand for $a \cdot b$.

Definition 2.1 (compact semirings). $(R, +, \cdot)$ is a compact semiring if,

(i) R is a compact Hausdorff space,

(ii) + and $\cdot$ are two associative operations on R,

(iii) + and $\cdot$ are continuous on the product space $R \times R$ taken with the usual product topology, and

(iv) the two identities

$$a(b + c) = ab + ac$$

$$(b + c)d = bd + cd \quad \text{hold on } R.$$

We denote the various parts of the additive and multiplicative semigroups by their usual symbols followed by a + or $\cdot$ in square brackets. For example, E [+] is the set of additive idempotents and K [$\cdot$] is the multiplicative minimal ideal. Of course E [+] is a multiplicative ideal, due to the two distributive laws, and so $K [\cdot] \subseteq E [+]$.

In all work that follows + does not denote an abelian operation unless this is specifically stated.

Definition 2.2 (compact semimodules over compact semirings). If $(M,+)$ is a compact semigroup, $(R, +, \cdot)$ is a compact semiring, and there is a continuous function $R \times M \to M$ (denoted by juxtaposition), such that the following identities are satisfied,

(i) $(r_1 r_2) x = r_1 (r_2 x)$ (R acts on M),

(ii) $(r_1 + r_2) x = r_1 x + r_2 x$, and

(iii) $r (x_1 + x_2) = r x_1 + r x_2$,

then M is a compact semimodule over the compact semiring R.

We note that our definition of compact semiring is that of Selden [25], who pioneered work in this field, and that the same notion has been vigorously pursued by Pearson [21], [22]. In fact a number of the results presented here are generalizations of their work.

The definition of semimodule is due to Wallace [35], and other results in this chapter are generalizations of suggestions made by him.

Definition 2.3 (compact nearrings). In this dissertation, this is to mean a structure $(R, +, \cdot)$ which is like a semiring in that it satisfies all the requirements for a semiring except the second distributive law. However, it is also supposed that $(R,+)$ is a group with neutral element 0, and that the additional property $0 x = 0$, is satisfied for every x in R.

2. Compact Semirings Which Are Multiplicatively Simple

Remarks. It is clear that, by using left trivial or right

trivial addition, any (compact) topological semigroup may be taken as the multiplication of a (compact) topological semiring. In particular, the multiplication may be any compact simple semigroup. Moreover, by taking the cartesian product of two compact simple semigroups and giving one left trivial addition and the other right trivial addition, the corresponding semiring is again a compact semiring which is both $(\cdot)$— and $(+)$— simple. We show in Corollary 2.6 below that every compact simple semiring which is both $(\cdot)$— and $(+)$— simple is obtainable in this way.

We begin by stating a theorem which we obtained by noting that a result of A. D. Wallace [36] was true under much less stringent hypotheses than he gave. We omit the proof because it is almost the same as his proof.

Theorem 2.4. Let $(R, +, \cdot)$ be a topological semiring with a multiplicative idempotent e, and such that $(R,+)$ is a rectangular band. Then $(R, +, \cdot)$ is iseomorphic to the cartesian product of topological semirings $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$, where the addition in R_1 is left trivial and the addition in R_2 is right trivial. $R_1 = R + e$, $R_2 = e + R$, and the map is $r \to (r + e, e + r)$.

The next theorem and its corollary can be obtained as corollaries to the main part of this last theorem, but we prove them separately because this is easy to do, and because the map given, hence the proof, is of a different character to that for the above theorem.

Theorem 2.5. Let $(R, +, \cdot)$ be a compact semiring in which $(R,+)$ is a rectangular band. Then $(R, +, \cdot)$ is iseomorphic to the cartesian product of compact semirings $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$, where the addition in R_1 is left trivial and the addition in R_2 is right trivial.

Proof. Choose $e^2 = e \in R$ (possible since R is compact). Then $e + e = e$ as well, because we have $(R,+)$ a band. By the theory of A. D. Wallace [31], for compact simple semirings, $R + e$ and $e + R$ are respectively minimal left and minimal right ideals of $(R,+)$. In these, addition is left trivial and right trivial respectively. Moreover, by the same theory, $(R_1 \times R_2, +)$ is iseomorphic to $(R,+)$ via the map $(a,b) \xrightarrow{\theta} a + b$. We define $(a,b) \cdot (c,d) = (ac, bd)$, and claim that $(a,b) \mapsto a + b$ is a multiplicative morphism as well.

$$\begin{aligned} \theta((a,b) \cdot (c,d)) &= \theta((ac, bd)) \\ &= ac + bd \\ &= ac + (ad + bc) + bd \quad \text{(by Kimura [15],} \end{aligned}$$

since $(R,+)$ is a rectangular band),

$$\begin{aligned} &= (a + b) \cdot (c + d) \\ &= \theta((a,b)) \cdot \theta((c,d)). \end{aligned}$$

Thus we are done.

<u>Corollary 2.6</u>. Let $(R, +, \cdot)$ be a compact semiring which is both $(\cdot)$— and $(+)$— simple. Then it is iseomorphic to a semiring obtained as explained in the Remarks above.

<u>Proof</u>. In this case the hypotheses of the last theorem are satisfied because $R \subseteq K[\cdot] \subseteq E[+] \subseteq R$, which gives $E[+] = R$, so that $(R,+)$ is a rectangular band.

<u>Theorem 2.7</u>. Let $(R, +, \cdot)$ be a compact semiring in which $(R,\cdot)$ is simple. Fix $e' \varepsilon E[\cdot]$. For $x, y \varepsilon R$, represented uniquely as

$$x = e_1 \, a \, e_2 \quad \text{and} \quad y = f_1 \, b \, f_2 \ , \quad \text{where}$$

$e_1, f_1 \varepsilon E[\cdot] \cap R e'$ and $e_2, f_2 \varepsilon e' R \cap E[\cdot]$ and,

$$a, b \ \varepsilon \ G = e' R e', \quad \text{then}$$

$\exists g_1 \varepsilon E[\cdot] \cap R e'$ and $g_2 \varepsilon e' R \cap E[\cdot]$, such that

$$e_1 \, a \, e_2 + f_1 \, b \, f_2 = g_1 \, (a + b) \, g_2.$$

<u>Proof</u>. $e_1 \, a \, e_2 + f_1 \, b \, f_2 = g_1 \, d \, g_2$ for some $g_1 \varepsilon E[\cdot] \cap R e'$, $g_2 \varepsilon e' R \cap E[\cdot]$ and $d \varepsilon G = e' R e'$.

Thus $e'(e_1 \, a \, e_2 + f_1 \, b \, f_2) \, e' = e'(g_1 \, d \, g_2) \, e'$.

So $e' \, e_1 \, a \, e_2 \, e' + e' \, f_1 \, b \, f_2 \, e' = e'(g_1 \, d \, g_2) \, e'$, by the associative and both distributive laws. Now as $e_1, f_1, g_1 \varepsilon R e'$,

$\therefore \quad R g_1 = R e_1 = R f_1 = R e' \ni e' = e'^2$.

Then $e' g_1 = e' e_1 = e' = e' f_1$. Also, as $e_2, f_2, g_2 \varepsilon e' R$,

$\therefore$ $g_2 R = e_2 R = f_2 R = e' R \ni e' = e'^2$, and then,

$g_2 e' = e_2 e' = e' = f_2 e'$. Thus $e' a e' + e' b e' = e' d e'$,

$\therefore$ $a + b = d$, since $a, b, d \in e' R e'$.

Theorem 2.8. Let $(R, +, \cdot)$ be a compact semiring. Then if $E[\cdot] \cap K[\cdot]$ is a subsemiring of $(R, \cdot)$, it follows that $(E[\cdot] + E[\cdot]) \cap K[\cdot] \subseteq E[\cdot] \cap K[\cdot]$.

Proof. Let $e, f \in E[\cdot] \cap K[\cdot]$, such that $e + f \in K[\cdot]$. Fix $e' \in E[\cdot] \cap K[\cdot]$ and let g be the idempotent of $e R \cap R e'$. Then $e R = g R$ and $R g = R e'$, so $e g = g$, $g e = e$, $g e' = g$, and $e' g = e'$. So $e = g e = g e' e$, and similarly $f = h e = h e' f$. Thus $e + f = g e' e + h e' f$

$$= g_1 (e' + e') f_1$$

(for some $g_1 \in E[\cdot] \cap R e'$ and some $f_1 \in e' R \cap E[\cdot]$, by Theorem 2.7 above)

$$= g_1 e' f_1$$

$$= g_1 f_1$$

$$= t \in E[\cdot] \quad \text{(by hypothesis).}$$

Theorem 2.9. Let $(R, +, \cdot)$ be a compact semiring in which $(R, \cdot)$ is simple and $(R, +)$ is simple. Choose $e^2 = e \in R$. Then denoting $E[\cdot] \cap R e$ by R_1, $e R \cap E[\cdot]$ by R_3, and $e R e$ by R_2, and considering the product space $R_1 \times R_2 \times R_3$ with the canonical product topology, we define the following two operations on the product.

(1) mult. $(e_1,g_1,f_1) \circ (e_2,g_2,f_2) = (e_1,g_1(f_1\ e_2)g_2,f_2)$

(2) addit. $(e_1,g_1,f_1) + (e_2,g_2,f_2) = (e_1 + e_2, g_1 + g_2, f_1 + f_2)$.

Then these definitions make sense and we may map our product onto R by the map $(e,g,f) \mapsto e\ g\ f$, which is an iseomorphism between the $(\cdot)$– operations and a homomorphism between the $(+)$– operations. Hence our map is a semiring iseomorphism.

Proof. We get immediately from Wallace [31] that (1) is well defined and is an iseomorphism between the $(\cdot)$– operations. We must show that $e_1 + e_2$ is again in $E[\cdot] \cap R\ e$, and that $f_1 + f_2$ is again in $e\ R \cap E[\cdot]$. Considering $R\ e$, this is a compact subsemiring of R, and satisfies the hypothesis of the previous theorem, so $e_1 + e_2$ is again in $E[\cdot] \cap R\ e$, and similarly for the other by considering $e\ R$. (We note that the ideas involved in the previous theorem and just here are generalizations of work of Pearson [22].) It remains to show that our map is an additive homomorphism.

$$\begin{aligned}(e_1 + e_2)\ (g_1 + g_2)\ (f_1 + f_2) &= e_1g_1f_1 + (\ldots.) + e_2g_2f_2 \\ &= e_1g_1f_1 \quad + \quad e_2g_2f_2\end{aligned}$$

by Kimura [15], since $(R,+)$ is a rectangular band.

Theorem 2.10. Let $(R, +, \cdot)$ be a compact semiring in which $(R,\cdot)$ is simple. Then denoting the structure decomposition of $(R,+)$ into maximal rectangular bands by $(R_\gamma,\ \gamma \in \Gamma)$ (Kimura [15]), it follows that this is a partition formed by a closed semiring congruence on R. Moreover, each $(R_\gamma, +, \cdot)$ is a compact subsemiring of R which is a union

of maximal $(\cdot)$— groups and which is $(\cdot)$— and $(+)$— simple. (As special cases we get: (i) $K\,[+]$ the minimal additive ideal of R, and (ii) the complement of any maximal additive ideal,--see remarks following the proof of this theorem.)

Proof. Define the relation τ by $x \,\tau\, y$ if, and only if, $x + y + x = x$ and $y + x + y = y$. Kimura has shown that this is an additive congruence and it induces our partition. We show that it is also a $(\cdot)$— congruence and that it is closed. Let $(x,y) \in \tau$ and let $q \in R$. Then $q\,x + q\,y + q\,x = q\,x$, using the left distributive law. Also $x\,q + y\,q + x\,q = x\,q$, using the right distributive law. Similarly, $q\,y + q\,x + q\,y = q\,y$ and $y\,q + x\,q + y\,q = y\,q$, so $(q\,x,\ q\,y),\ (x\,q,\ y\,q) \in \tau$. Let $(s,\ t) \in R \times R \setminus \tau$. Then either $s + t + s \neq s$ or $t + s + t \neq t$. If $s + t + s \neq s$, then $\exists\, U(s),\ U(t)$, open sets about s and t respectively, such that

$$U(s) + U(t) + U(s) \cap U(t) = \phi$$

(from Hausdorff space and continuity of addition).
Then $W((s,t)) = U(s) \times U(t) \subseteq R \times R \setminus \tau$, and is open. Otherwise, similarly consider $t + s + t = t$. Thus τ is closed, so each R_γ is closed, hence compact. Now since each maximal $(\cdot)$— group is contained in a τ class, and each R_γ is contained in R, which is simple, we conclude that each R_γ is a union of maximal $(\cdot)$— groups. Now the product of a pair of maximal $(\cdot)$— groups is contained in another, so we need only show that $(E\,[\cdot] \cap R_\gamma) \cdot (E\,[\cdot] \cap R_\gamma) \subseteq R_\gamma$, and it then follows that $R_\gamma \cdot R_\gamma \subseteq R_\gamma$.

Let $e, f \in E[\cdot] \cap R_\gamma$.

Then $$e\,f + f + e\,f = e\,f + f\,f + e\,f$$
$$= (e + f + e)\,f$$
$$= e\,f \quad , \text{ and}$$
$$f + e\,f + f = f\,f + e\,f + f\,f$$
$$= (f + e + f)\,f$$
$$= f\,f$$
$$= f \quad .$$

Thus $e\,f \;\tau\; f$ and so $e\,f \in R_\gamma$. Finally, we need only remark that any closed subsemigroup of $(R,\cdot)$ will be simple.

As indicated in parentheses, Theorem 2.10 subsumes earlier work by the author [23], where the minimal ideal $K[+]$ and the complement of any maximal $(+)$– ideal were each separately shown to be subsemirings when $(R,\cdot)$ is simple. This comes from Lemma 2.11, Theorem 2.13, and Corollary 2.15 below.

Lemma 2.11. If $(R,+)$ is a semigroup, and I any ideal of R, then for any rectangular band B, a subsemigroup of R, either $B \subseteq I$, or B does not meet I.

Proof. If $x \in I \cap B$ and y is any element of B, then $y + x + y = y$, which is in I, since I is an ideal.

Comment 2.12. In particular Lemma 2.11 applies to the minimal ideal $K[+]$ and to any maximal ideal.

We need the following result.

Theorem 2.13 (Faucett-Koch-Numakura [8]). If $(S,+)$ is a compact mob in which J is a maximal ideal, and if also $S \setminus J$ is a union of (additive) groups, then $S \setminus J$ is a subsemigroup of S which is compact and simple.

Corollary 2.14. If $(R, +, \cdot)$ is a compact semiring in which $(R,\cdot)$ is simple, then the complement of any maximal additive ideal $J\ [+]$, is a subsemigroup of $(R,+)$ and is compact and simple.

Proof. $E\ [+] = R$, because $(R,\cdot)$ is simple, and then by Theorem 2.13 we are done.

This means that since $K\ [+]$ and the complement of any maximal $(+)$– ideal are rectangular subbands of $(R,+)$ when $(R,\cdot)$ is simple, thus they must be maximal rectangular subbands of $(R,+)$. (Comment 2.12.)

A corollary to a result of Pearson [22] gives us the following product theorem for R/τ.

Theorem 2.15. Choose any element $\bar{e} \in R/\tau$. Then R/τ is iseomorphic to the cartesian product of the semirings

$$R_1 = (R/\tau)\,\bar{e}\ , \qquad R_2 = \bar{e}\ (R/\tau)$$

via the map $(a,b) \mapsto ab$. Thus R/τ may only come as the product of a

pair of additive bands, with left trivial multiplication in one, and right trivial multiplication in the other.

3. A Method for Constructing Some Compact Semirings Which Are $(\cdot)$-simple

Theorem 2.16. If $(X,+)$ and $(Y,+)$ are compact bands, and if $(G, +, \cdot)$ is a compact semiring with $(G,\cdot)$ a group, then any continuous function $\sigma\colon Y \times X \to G$ (denoted by juxtaposition where no confusion is likely) which has the properties:

(i) $(y_1 + y_2)x = y_1x + y_2x$

(ii) $y(x_1 + x_2) = yx_1 + yx_2$,

gives rise to a compact semiring which is $(\cdot)$-simple. (We take $R = X \times G \times Y$ with the usual topology, and define

$$(x,g,y) \cdot (x',g',y') = (x,g\sigma((y,x'))\, g',y') \quad \text{and}$$

$$(x,g,y) + (x',g',y') = (x+x',g+g',y+y') .)$$

Proof. That $(R,\cdot)$ is a compact mob follows from Wallace [31]. $(R,+)$ is merely the product of three bands, and so is a band mob. We need only check the distributive laws.

$$\begin{aligned}
&(x,g,y)\ [(x_1,g_1,y_1) + (x_2,g_2,y_2)] \\
&= (x,g,y)\ [(x_1 + x_2,\ g_1 + g_2,\ y_1 + y_2)] \\
&= (x,g\sigma((y,x_1 + x_2))\ (g_1 + g_2),\ y_1 + y_2) \\
&= (x,g[\sigma((y,x_1)) + \sigma((y,x_2]\ (g_1 + g_2),\ y_1 + y_2) \quad \text{(by (i))} \\
&= (x,g\sigma((y,x_1))\ g_1 + \ldots + g\sigma((y,x_2))\ g_2,\ y_1 + y_2) \\
&= (x,g\sigma((y,x_1))\ g_1 + g\sigma((y,x_2))\ g_2,\ y_1 + y_2) \quad \text{(Kimura [15])}.
\end{aligned}$$

(Pearson [21] has shown that $(G,+)$ is a rectangular band.)

$$(x,g,y) \cdot (x_1,g_1,y_1) + (x,g,y) \cdot (x_2,g_2,y_2)$$
$$= (x,g\sigma((y,x_1))\ g_1,y_1) + (x,g\sigma((y,x_2))\ g_2,y_2)$$
$$= (x + x,\ g\sigma((y,x_1)\ g_1 + g\sigma((y,x_2))\ g_2,\ y_1 + y_2)$$
$$= (x,g\sigma((y,x_1)\ g_1 + g\sigma((y,x_2))\ g_2,\ y_1 + y_2)$$

(because X and Y are bands). Thus we have left distribution. The other distribution follows similarly using (ii) above.

Theorem 2.17. For any compact semiring $(R, +, \cdot)$ we may define, on the minimal multiplicative ideal $K[\cdot]$, an addition $\oplus$, and multiplications σ_g for each g in a fixed $H[\cdot](e) \subseteq K[\cdot]$, such that $(K[\cdot], \oplus, \sigma_g)$ is a compact semiring with simple multiplication. When $g = e$, we obtain the original multiplication, and if $K[\cdot] \subseteq E[\cdot]$, and $(K[\cdot], +, \cdot)$ is a subsemiring of R, then $\oplus$ is the same as $+$.

Proof. For any fixed $e^2 = e \in K[\cdot]$, $H[\cdot](e) = e\,K[\cdot]\,e$ is a compact semiring under $+$ and $\cdot$, and is a $(\cdot)$– group. Also, $E[\cdot] \cap K[\cdot]\,e$ and $e\,K[\cdot] \cap E[\cdot]$ are compact semirings under $+$ and $\cdot$ (see Theorem 2.9 above). More importantly, these last two sets are $(+)$– bands, since they are included in $K[\cdot] \subseteq E[+]$. Finally, since we are in a semiring, if we select $g \in e\,K[\cdot]\,e$, then the map $(y,x) \overset{\sigma_g}{\mapsto} y\,g\,x$ from $(e\,K[\cdot] \cap E[\cdot]) \times (E[\cdot] \cap K[\cdot]\,e) \to e\,K[\cdot]\,e$ is continuous, and has the properties

$$\sigma_g((y,\ x_1 + x_2)) = \sigma_g((y,\ x_1)) + \sigma_g((y,\ x_2))$$

and $\sigma_g((y_1 + y_2, x)) = \sigma_g((y_1, x)) + \sigma_g((y_2, x))$.

We know from Wallace [31] that the space $(E[\cdot] \cap K[\cdot]e) \times eK[\cdot]e \times (eK[\cdot] \cap E[\cdot])$ with multiplication $(a,b,c) \cdot (a',b',c') = (a,b\sigma_g((c,a'))b',c')$, is a compact simple mob. Moreover, when $g = e$, it is iseomorphic to $(K[\cdot], \cdot)$ by the map $(a,b,c) \to a\,b\,c$. Thus this map is always a homeomorphism of our product to $K[\cdot]$. Now, by applying Theorem 2.16 above, we may take componentwise addition for $\oplus$, and then, via the homeomorphism, $(K[\cdot], \oplus, \sigma_g)$ is a compact semiring. We have already noted that when $g = e$, σ_g corresponds to $\cdot$, and if $K[\cdot] \subseteq E[\cdot]$, and $K[\cdot]$ is a subsemiring of R, then Pearson [22] has already shown that our addition corresponds to +.

We do not know whether compact semirings with simple multiplication are more complicated than this, but, due to our inability to produce a semifactorization theorem corresponding to our construction, it is suspected that they are more complicated. It is not known, for compact connected semirings in the plane, whether or not $K[\cdot]$ has to be a subsemiring, but once again it is conjectured that it need not be.

4. Double T-ideals in Compact Semirings

Let $(R, +, \cdot)$ be a compact semiring, and T a closed subsemiring of R. Then a $(\cdot)$—T-ideal of R, is any subset $\phi \neq A \subseteq R$, such that $TA \cup AT \subseteq A$. A $(+)$—T-ideal is similarly defined. Then a double

T-ideal of R, is a set A, which is both a $(\cdot)$-- and $(+)$-T-ideal of R.

For the case T = R, Selden [25] has shown the existence of a minimal double R ideal which is, moreover, unique and closed. A similar proof gives the existence of a unique minimal double T-ideal which is closed. We have not found any reference in the literature to the question of proper double T-ideals being contained inside maximal proper double T-ideals. That this is so, and that they are open, will be shown shortly. This result, or the lemma which precedes it, is used in several places in what follows in this dissertation so it is quite important. First we make some machinery to save tedious chains of symbols in these arguments and others later. We note that this next theorem has no apparent counterpart in Universal BATS (Section 2.5), and this remark applies to all theorems which depend on it. It is of course a generalization of Koch and Wallace [16], and Wallace [32][30].

Theorem 2.18. For $(R, +, \cdot)$ a compact semiring, and T a closed subsemiring of R, if $R \supseteq A \neq \phi$, consider the set

$P_T = P_T(A) = \{T_i + T_j A T_k + T_\ell : i,\ell \in \{1,2\}\ j,k \in \{2,3\}\}$, where T_1 is a formal additive identity on R, $T_2 = T$, and T_3 is a formal multiplicative identity on R. Then P_T is the smallest double T-ideal of R which contains A.

Proof. $T\,(T_i + T_j A T_k + T_\ell) \subseteq T\,T_i + T\,T_j A T_k + T\,T_\ell$ (left distributive law). Now if i = 1, or ℓ = 1, then the respective terms could be omitted in the first line, which means that we can, for all

values of i and ℓ, write $\subseteq T_i + T\, T_j\, A\, T_k + T_\ell$. Now also, whether $j = 2$ or 3, $T\, T_j \subseteq T_j$, so we can put $\subseteq T_i + T_j\, A\, T_k + T_\ell$. It then follows that $T\, P_T \subseteq P_T$. By similar arguments we get the remaining necessary inclusions, $P_T\, T \cup (T + P_T) \cup (P_T + T) \subseteq P_T$. Thus P_T is certainly a double T-ideal containing A (take $i = 1$, $j = 3$, $k = 3$, $\ell = 1$). Moreover, any double T-ideal containing A must contain P.

Lemma 2.19. Let $(R, +, \cdot)$ be a compact semiring, T a closed subsemiring of R, K(R) the minimal double T-ideal of R, and U an open subset of R, which contains K(R). Then the union, J, of all double T-ideals contained in U, is an open subset of R. (Moreover, either J is closed, or J^* meets the boundary of U.)

Proof. We choose $x \in J$, and then consider $P_T(\{x\}) = \cup\{T_i + T_j\{x\}\, T_k + T_\ell\}$ as in Theorem 2.18 above. Then $P_T(x) \subseteq J \subseteq U$. By several applications of Wallace's Theorem we obtain an open set $V(x)$ such that $P_T(V) \subseteq U$. Moreover, $P_T(V)$ is a double T-ideal containing $V(x)$ from Theorem 2.18 above. So we have that $x \in V(x) \subseteq P_T(V) \subseteq J$. Since x was arbitrary, J is open. Now, since J^* is also a double T-ideal, it follows that either $J^* = J$, or $J^* \setminus J$ is included in (complement of U) $\cap\ U^* \subseteq$ boundary of U.

Theorem 2.20. If $(R, +, \cdot)$ is a compact semiring, if T is a closed subsemiring or R, and I and is any proper double T-ideal of R, then I is contained in a maximal proper double T-ideal of R, which is open. Any maximal proper double T-ideal of R will be open.

Proof. If $x \in R \setminus I$, then $R \setminus \{x\}$ is an open subset of R which contains I. Moreover, I contains K(R) the minimal double T-ideal of R. So, applying Lemma 2.19, we get an open proper double T-ideal of R which contains I. Let $\mathcal{J}$ be the set of all possible open proper double T-ideals of R which contain I. Then, ordering by $J_1 \leq J_2$ if $J_1 \subseteq J_2$, and considering a chain, it is clear due to the compactness of R, that the union of the chain is not R. Also, clearly it is open and a double T-ideal of R which contains I. Then, by Zorn's Lemma, we obtain a double T-ideal J, maximal in $\mathcal{J}$. We must finally show that it is truly a maximal double T-ideal.

If $R \supsetneq J' \supsetneq J \supseteq I$, J' a double T-ideal of R, then by repeating the argument so far (or using Lemma 2.19 again), we would get J'' a proper double T-ideal of R, open and containing J', hence containing I. However, since $J'' \supseteq J' \supsetneq J$, this would contradict the maximality of J in $\mathcal{J}$.

Clearly any maximal proper double T-ideal of R will be open.

5. Compact Universal BATS

Definition 2.21. A compact space S, with a set Γ_S of binary operations on it, each of which is continuous on the product space, will be called a compact universal BAT. Strictly speaking, we need Γ_S to be a set of elements such that although distinguishable by a "marker" in some way from each other, elements of some subset of Γ_S may, as binary operations on S, have the same answer in S for every

pair in S x S. It may be that certain identities are satisfied on S by certain subsets of Γ_S. For example, on singleton subsets we might have the associativity identity. On subsets of Γ_S of cardinality two, we may have some distributivity identity. A particular compact universal BAT may be denoted by (S, Γ_S). If (S, Γ_S) and (T, Γ_T) are compact universal BATS, and there is a 1 - 1 and onto function $g: \Gamma_S \to \Gamma_T$ with the property that for each $\gamma \in \Gamma_S$ any identity involving γ will also be present in Γ_T involving $g(\gamma)$ in the same role, then we say that (S, Γ_S) and (T, Γ_T) are of the <u>same type relative</u> to g.

If (S, Γ_S) and (T, Γ_T) are of the same type relative to g, and if $\phi: S \to T$ is a continuous function with the property that, for each $s_1, s_2 \in S$, and for each $\gamma_S \in \Gamma_S$, $\phi(\gamma_S((s_1,s_2))) = g(\gamma_S)((\phi(s_1), \phi(s_2)))$, then we say that ϕ, with reference to g, is a universal BAT morphism between (S, Γ_S) and (T, Γ_T).

If ~ is a closed equivalence on S, with the property that $(x,y)\in\sim$ implies that, for each $\gamma_S \in \Gamma_S$, and for each $s \in S$, we have

$$(\gamma_S((s,x)), \gamma_S((s,y))) \quad \text{and} \quad (\gamma_S((x,s)), \gamma_S((y,s)))$$

each also in ~, then we call ~ a closed universal BAT congruence relative to Γ_S. It is well known that if we define $\gamma_{S/\sim}(([s_1],[s_2])) = [\gamma_S((s_1,s_2))]$ for each $[s_1]$ and $[s_2]$ in $S/\sim$, then $\gamma_{S/\sim}$ is a continuous binary operation on $S/\sim$. Consider the set $\boxed{\Gamma_{S/\sim}} = \{\gamma_{S/\sim}: \gamma_S \in \Gamma_S\}$. Even though γ_S and γ_S' are "marked" differently in Γ_S, they may have the same effect on S x S, thus

$\gamma_{S/\sim}$ and $\gamma'_{S/\sim}$ would have the same evaluations on $S/\sim \times S/\sim$. However, this would not matter, as the "marks" on γ_S, and γ'_S, would transfer to $\gamma_{S/\sim}$, and $\gamma'_{S/\sim}$, respectively, if we agree to distinguish elements of $\boxed{\Gamma_{S/\sim}}$ not by their evaluation of $S/\sim \times S/\sim$, but by their method of definition. This also removes the worry that, even when two elements γ_S, γ'_S of Γ_S have <u>different</u> evaluations on $S \times S$, it could still be that $\gamma_{S/\sim}$ and $\gamma'_{S/\sim}$ have the same evaluation on $S/\sim \times S/\sim$. Having said this we remove the box and simply write $\Gamma_{S/\sim}$. By what we have said, there is now a clear 1 - 1 onto map $g'\colon \Gamma_S \to \Gamma_{S/\sim}$, and, (S, Γ_S) and $(S/\sim, \Gamma_{S/\sim})$ are of the same type relative to g'. It is also well known that, for each $\gamma_S \in \Gamma_S$, the map $\theta\colon S \to S/\sim$, given by $s \mapsto [s]$, has the property that for each $s_1, s_2 \in S$, $\theta(\gamma_S((s_1,s_2))) = g'(\gamma_S)((\theta(s_1), \theta(s_2)))$. So θ is, with reference to g', a universal BAT morphism between (S, Γ_S) and $(S/\sim, \Gamma_{S/\sim})$.

<u>Theorem 2.22 (Monotone--light factorization)</u>. Let (S, Γ_S) and (T, Γ_T) be compact universal BATS (of the same type relative to $g\colon \Gamma_S \to \Gamma_T$). Let θ, with reference to g, be a universal BAT morphism between (S, Γ_S) and (T, Γ_T), $\theta\colon S \to T$. Then there is a compact universal BAT $(S', \Gamma_{S'})$, a $g'\colon \Gamma_S \to \Gamma_{S'}$, and a $g''\colon \Gamma_{S'} \to \Gamma_T$, such that: (i) (S, Γ_S) and $(S', \Gamma_{S'})$ are of the same type relative to g', (ii) $(S', \Gamma_{S'})$ and (T, Γ_T) are of the same type relative to g'', and (iii) $g(\cdot) = g'' \circ g'(\cdot)$. Moreover, there are universal BAT morphisms $\theta'\colon S \to S'$ relative to g', and $\theta''\colon S' \to T$ relative to g'', such that

θ' is onto S', and,

(a) $\theta'^{-1}(s)$ is connected for all $s \in S'$,

(b) $\theta''^{-1}(t)$ is totally disconnected for each $t \in \theta''(S')$, and,

(c) The diagram

$$\begin{tikzcd} S \arrow[d, "\theta'"'] \arrow[dr, "\theta"] & \\ S' \arrow[r, "\theta''"'] & T \end{tikzcd}$$

commutes.

Proof. Define $\sim$ on S by, $(x,y) \in \sim$ if and only if x and y are in the same component $\bar{C}_x$ of $\theta^{-1}(\theta(x))$. It is clear that $\sim$ is an equivalence. Then for each $s \in S$, and for each $\gamma_S \in \Gamma_S$, ${}_sA = \{\gamma_S((s,y)) \colon y \in \bar{C}_x\}$, and $A_s = \{\gamma_S((y,s)) \colon y \in \bar{C}_x\}$, both being continuous images of the connected set $\bar{C}_x$, are connected. If $q \in \bar{C}_x$, then $\theta(q) = \theta(x)$, so that for each $s \in S$, and for each $\gamma_S \in \Gamma_S$, we have

$$\begin{aligned} \theta(\gamma_S((s,q))) &= g(\gamma_S)((\theta(s),\ \theta(q))) \\ &= g(\gamma_S)((\theta(s),\ \theta(x))) \\ &= \theta(\gamma_S((s,x))). \end{aligned}$$

Thus ${}_sA \subseteq \theta^{-1}(\theta(\gamma_S((s,x))))$. Now ${}_sA$ is connected, so that ${}_sA \subseteq \bar{C}_{\gamma_S((s,x))}$, and so, if $(y_1, y_2) \in \sim$, then $(sy_1, sy_2) \in \sim$. Similarly, $A_s \subseteq \bar{C}_{\gamma_S((x,s))}$, and $(y_1s, y_2s) \in \sim$, and so $\sim$ is a congruence on S. We must show now that $\sim$ is closed as a subset of $S \times S$, or equivalently, that $S \times S/\sim$ is open. We show that $S \times S/\sim$ is open.

Let $x \not\sim y$. Then either (i) $\theta(x) \nparallel \theta(y)$, in which case there certainly exist open sets $U(x)$, $V(y) \ni \theta(U(x)) \cap \theta(V(y)) = \phi$ by continuity of θ, giving $U(x) \times V(y) = W(x,y) \subseteq S \times S/\sim$, and $W(x,y)$ is an open set in $S \times S$ which contains (x,y), or (ii) $\theta(x) = \theta(y)$ but $y \notin \bar{C}_x$. In this case $\exists$ relatively open sets $\bar{U}_x$, $\bar{U}_y$, with union $\theta^{-1}\theta(x)$, and intersection ϕ, such that $x \in \bar{U}_x$ and $y \in \bar{U}_y$. Since these sets are also relatively closed, hence closed, $\exists$ open sets in S, $\tilde{\bar{U}}_x$, $\tilde{\bar{U}}_y$, with empty intersection, which contain $\bar{U}_x$ and $\bar{U}_y$ respectively, and which (using a net argument on the space of compact subsets of S) do not both meet any one $\sim$ class outside of $\bar{C}_x$). These must, by an elementary set theoretic argument, have the properties

$$\tilde{\bar{U}}_x \cap \theta^{-1}\theta(x) = \bar{U}_x \quad \text{and}$$

$$\tilde{\bar{U}}_y \cap \theta^{-1}\theta(x) = \bar{U}_y \quad .$$

Then, $\tilde{\bar{U}}_x \times \tilde{\bar{U}}_y \subseteq S \times S\backslash\sim$ because, if $(a,b) \in \bar{U}_x \times \bar{U}_y$, then either $\theta(a) \nparallel \theta(b)$, or, if $\theta(a) = \theta(b)$, then a and b are not in the same component of $\theta^{-1}\theta(x)$ since they are in opposite parts of a separation of $\theta^{-1}\theta(x)$. Thus $\sim$ is closed. Letting $(S', \Gamma_{S'}) = (S/\sim, \Gamma_{S/\sim})$, and defining $\theta'\colon S \to S' = S/\sim$ by $\theta'(s) = [s]$, we have that θ' is a universal BAT morphism. Moreover, it is clear that θ' is onto S', and that, for each $s \in S'$, $\theta'^{-1}(s)$ is connected. We define $g''\colon \Gamma_{S'} = \Gamma_{S/\sim} \to \Gamma_T$ by $g''(\gamma_{S/\sim}) = g(g'^{-1}(\gamma_{S/\sim})) = g(\gamma_S)$. Now, by the earlier remarks in Definition 2.21, g'' is well defined, 1 - 1, and onto, and moreover, $g(\cdot) = g'' \circ g'(\cdot)$. We thus have that $(S', \Gamma_{S'})$ and (T, Γ_T) are universal BATS of the same type with respect to g''. We define

$\theta''([s]) = \theta(s)$, and by Sierpinski's Lemma, θ'' will be a universal BAT morphism relative to g''. It now remains only to show that, for each $t \in \theta''(S')$, we have $\theta''^{-1}(t)$ is totally disconnected. Let p and q be in $\theta''^{-1}(t)$, with $p \nmid q$. U_p and U_q lie in distinct sides of a separation of $\theta^{-1}(t)$, $\bar{U}|\bar{V}$, say $U_p \subseteq \bar{U}$. Moreover, for each $y \in \theta''^{-1}(t)$, U_y must lie either in $\bar{U}$ or $\bar{V}$. Thus we may divide $\theta''^{-1}(t)$ into two sets A and B, one of which, A, has elements p' which, like p, have $U_{p'} \subseteq \bar{U}$, and the other, B, has elements $q' \nmid U_{q'} \subseteq \bar{V}$. Clearly, $A = \theta'(\bar{U})$ and $B = \theta'(\bar{V})$. As $\theta^{-1}(t)$ is closed in S, so are $\bar{U}$ and $\bar{V}$.

Thus, $A|B$ is a separation of $\theta''^{-1}(t)$, with $p \in A$, and $q \in B$, and so $\theta''^{-1}(t)$ is totally disconnected.

Theorem 2.23. Let (S, Γ_S) be a compact universal BAT, and Δ the diagonal of $S \times S$. For $A \subseteq S \times S$, let $\tilde{A}$ be the intersection of all closed equivalences on S, which contain A. Then if, for each $\gamma_S \in \Gamma_S$, for each $(x,x) \in \Delta$ and for each $(p,q) \in A$, we have that,

$$\gamma_S \times \gamma_S(((x,x),\ (p,q))) = (\gamma_S((x,p)),\ \gamma_S((x,q)) \in A,$$

and

$$\gamma_S \times \gamma_S(((p,q),\ (x,x))) \in A$$

then A is the smallest closed universal BAT congruence on S relative to Γ_S.

Proof. By Hofmann and Mostert [12, p. 22] the result is true of a single operation, and our hypotheses contain what is needed for each separate operation.

Corollary 2.24. If we have a compact semiring $(R, +, \cdot)$, then for $X \subseteq R \times R$, the minimal subset Q of $R \times R$ with the properties:

(i) $Q \supseteq X$ and

(ii) Q is a double Δ-ideal,

is given by $P_\Delta(X)$ in Theorem 2.18, so then $(P(X))^{\sim}$, the minimal closed equivalence on R which contains $P_\Delta(X)$, is the minimal closed semiring congruence on R which contains X.

Theorem 2.25. Let (S, Γ_S) be a compact universal BAT, and $J = \{R(i)\colon i \in I\}$ be a collection of closed universal BAT congruences with respect to Γ_S, with intersection Δ, and which have the property that, for each $R(i_1), R(i_2) \in J$, $\exists$ an $R(i_3) \in J$, such that $R(i_3) \subseteq R(i_1) \cap R(i_2)$. Then, if $R(i_2) \subseteq R(i_1)$, there is a canonical universal BAT iseomorphism $\pi_{i_1 i_2}\colon S/R(i_2) \to S/R(i_1)$, and if $R_k \subseteq R_j \subseteq R_i$, then

$$\pi_{ij} \circ \pi_{jk}(\cdot) = \pi_{ik}(\cdot).$$

Finally, $\underset{\leftarrow}{\lim}\,(S/R(i), \pi_{\ell m}, J)$ is a universal compact BAT with respect to the operations Γ_S, and is iseomorphic to (S, Γ_S). (Here we have used Γ_S, rather than $\Gamma_{S/R}$, and omitted reference to the maps $\Gamma_{S/R(i)} \to \Gamma_{S/R(j)}$, since no confusion seems likely.)

Proof. By Hofmann and Mostert [12, p. 49] the result is true for one operation and since the map for the iseomorphism is the same

regardless of the operation involved the theorem follows.

Corollary 2.26. The theorem is true for a semiring.

Definition 2.27. Let (S, Γ_S) be a compact universal BAT. A metric on S is called subinvariant if, for each $\gamma_S \varepsilon \Gamma_S$, and for each triple a,x,y in S, we have $d(\gamma_S((a,x)), \gamma_S((a,y))) \leq d(x,y)$ and $d(\gamma_S((x,a)), \gamma_S((y,a))) \leq d(x,y)$.

6. Concerning Inverse Limits of Semirings

The next theorem which we prove is conjectured to be false in the general setting of compact universal BATS, although an examination of the proof will reveal that it could be proved in cases where there are certain identities between elements of Γ_S other than the particular situation that we deal with. Once again the proof hinges on distributivity in a semiring. Thus it is conjectured that the result is false in some cases where cardinality of Γ_S is two, but at present we do not have a counter example. We begin with some lemmas, and remark that the approach was suggested to us by Professor Sigmon who modified analogous work by Hofmann and Mostert [12], which relied on a result in Kelley [14].

Lemma 2.28. Suppose that $(R, +, \cdot)$ is a compact semiring and that $\{U_n: n = 0,1,2,\ldots\}$ is a sequence of subsets of $R \times R$ with $U_0 = R \times R$ and satisfying, for each $n \geq 1$,

(i) $\Delta \subseteq U_n^\circ \subseteq U_n = \overline{U}_n$

(ii) $U_n \circ U_n \circ U_n \subseteq U_{n-1}$

(iii) $U_n^{-1} = U_n$

(iv) $\Delta U_n \cup U_n \Delta \cup (\Delta + U_n) \cup (U_n + \Delta) \subseteq U_n$.

If $\bigcap_{n=1}^{\infty} U_n = \Delta$, then R admits a subinvariant metric.

Proof. We define $f: R \times R \to$ Reals by,

$$f(x,y) = 1/2^n, \quad (x,y) \in U_n/U_{n+1}$$
$$= 0 \quad , (x,y) \in \Delta ,$$

and note that $(x,y) \in U_n$, if and only if, $f((x,y)) < 1/2^{n-1}$. Then define $d: R \times R \to$ Reals by

$$d(x,y) = \inf \left\{ \sum_{i=0}^{k-1} f(x_i,x_{i+1}) \colon x_0,\ldots,x_k \in R,\ x_0 = x,\ x_k = y,\ k \in \{1,2,\ldots\} \right\}.$$

We show that d is a metric, and that

$$1/2\ f(x,y) \leq d(x,y) \leq f(x,y) .$$

$$d(x,y) = \inf \left\{ \sum_{i=0}^{k-1} f(x_i,x_{i+1}) \colon x_0,\ldots,x_k \in R,\ x = x_0,\ y = x_k,\ k \in \{1,2,\ldots\} \right\}$$
$$\leq f(x,y) ,$$

since we may choose $k = 1$, $x_0 = x$, $x_1 = y$.

In particular, $d(x,x) \leq f(x,x)$. From now on we do not write the set over which k varies, it is understood. Now, since we suppose that $\bigcup_{n=1}^{\infty} U_n = \Delta$, then, for each $n \geq 1$, $(x,x) \in U_n$, and so, for each $n \geq 1$, $0 \leq d(x,x) \leq f(x,x) < 1/2^{n-1}$, giving that $d(x,x) = 0$. Consider $x,y,t \in R$. Now $d(x,y) \leq \sum_{i=0}^{k-1} f(x_i,x_{i+1}) + \sum_{i=k}^{k+k'-1} f(x_i,x_{i+1})$ for <u>any</u> $x_0,\ldots x_{k+k'} \in R$, where $x_0 = x$, $x_k = t$, and $x_{k+k'} = y$.

$$\therefore \quad d(x,y) \leq d(x,t) + \sum_{i=k}^{k+k'-1} f(x_i,x_{i+1}).$$

$$\therefore \quad d(x,y) \leq d(x,t) + d(t,y).$$

Now we show that $1/2\ f(x,y) \leq d(x,y)$, or equivalently, that

$f(x,y) < 2 \sum_{i=0}^{m-1} f(x_i,x_{i+1})$ for all $x_0,\ldots,x_m \in R$ such that $x_0 = x$, and $x_m = y$. The proof is by induction on m. When $m = 1$ it is clearly true since $f(x,y) \leq 2\ f(x,y)$. Now, by assuming that it is true for all chains of length n, we must show it is true for $m = n + 1$. That is, if $x_0,\ldots,x_{n+1} \in R$, $x_0 = x$, and $x_{n+1} = y$, then we must show that

$f(x_0,x_{n+1}) \leq 2 \sum_{i=0}^{n} f(x_i,x_{i+1})$. We set ${}_r\ell_s = \sum_{i=r}^{s} f(x_i,x_{i+1})$, $0 \leq r \leq s \leq n$. We put $a = {}_0\ell_n$. Without loss of generality, let $k \in \{0,1,\ldots,n\}$ be the largest such that ${}_0\ell_k \leq a/2$, that is, ${}_0\ell_n \geq 2\ {}_0\ell_k$. (If this were not possible we could reargue from the other end of the chain.)

Now, since ${}_0\ell_k + {}_k\ell_{k+1} \geq a/2$, it follows that $a/2 \geq a - {}_0\ell_k - {}_k\ell_{k+1} = {}_{k+1}\ell_{n+1}$. Now $f(x_0,x_k) \leq 2\ {}_0\ell_k \leq 2\ a/2 = a$, and $f(x_{k+1},x_{n+1}) \leq 2\ {}_{k+1}\ell_{n+1}$ by our induction hypotheses. Also, $f(x_k,x_{k+1}) \leq a$. Let m be the least integer such that $2^{-m} \leq a$. Then (x_0,x_k), (x_k,x_{k+1}), and (x_{k+1},x_{n+1}), all belong U_m, so by hypothesis $(x_0,x_{n+1}) \in U_{m-1}$.

$\therefore \quad f(x_0,x_{n+1}) < 2^{-(m-1)} = 2 \times 2^{-m}$, and so we have that $f(x,y) \leq 2\ d(x,y)$, as required. Thus it is clear that

(*)— $U_{n+1} \subseteq \{(x,y):\ d(x,y) < 1/2^n\} = D_n \subseteq U_n$. If $x \neq y$, then $\exists U_n \ni (x,y) \notin U_m$, by $\bigcap_{n=1}^{\infty} U_n = \Delta$. Thus $f(x,y) \geq 1/2^{n-1}$ and so $d(x,y) \geq 1/2^n$. To have d a metric we still need $d(x,y) = d(y,x)$.

Consider $\sum_{i=0}^{n-1} f(x_i,x_{i+1})$ for $x_0,x_1,\ldots,x_n \in R$, where $x_0 = x$, and $x_n = y$.

Now $f(x_{i+1},x_i) = 1/2^n$, $(x_{i+1},x_i) \in U_n \setminus U_{n+1}$; $= 0$, $(x,y) \in \Delta$. As $U_{n+1}^{-1} = U_{n+1}$, and $U_n^{-1} = U_n$, it follows that if $x_i \neq x_{i+1}$, then $f(x_{i+1},x_i) = f(x_i,x_{i+1})$, and this is also true if $x_i = x_{i+1}$, so

$$\sum_{j=0}^{n-1} f(y_j,y_{j+1}) = \sum_{i=0}^{n-1} f(x_i,x_{i+1})$$

where $y_j = x_{n-j}$.

Thus $d(x,y)$ and $d(y,x)$ are each the infimum of the same set of real numbers and so are equal. Subinvariance follows from (iv), by using (*) above, since, if $x = y$, $\exists\ n \ni (x,y) \in U_n$ and $(x,y) \notin U_{n+1}$. Then

(ax,ay), (xa,ya), $(a + x,\ a + y)$, and $(x + a,\ y + a)$ all belong to U_n, and so $f(ax,ay) \leq f(x,y)$.

$$d(p,q) = \inf \left\{ \sum_{i=0}^{n} f(x_i,x_{i+1}) \colon x_0,\ldots,x_n \in R;\ x_0 = p,\ x_n = q \right\}$$

$$d(ap,aq) \leq \inf \left\{ \sum_{i=0}^{n} f(ax_i,ax_{i+1}) \colon x_0,\ldots,x_n \in R;\ x_0 = p,\ x_n = q \right\}$$

$$\leq \inf \left\{ \sum_{i=1}^{n} f(x_i,x_{i+1}) \colon x_0,\ldots,x_n \in R;\ x_0 = p,\ x_n = q \right\}$$

$$= d(x,y)\ .$$

Similarly, the other invariance properties may be proven. Let $B_{1/2^n}(p) = \{x\colon\ d(x,p) < 1/2^n\}$. Then

$$B_{1/2^n}(p) = \pi_1\ [D_n \cap (R \times \{p\})]$$

$$\supseteq \pi_1\ [U^\circ_{n+1} \cap (R \times \{p\})]$$

$$= \pi_1\ [(V = V^\circ) \times \{p\}].$$

Since, if $(x,p) \in U^\circ_{n+1}$, then $\exists\ U_x \ni U_x \times \{p\} \subseteq U^\circ_{n+1}$, and then

$$p \in V = \bigcup_{x \in \pi_1 (U^\circ_{n+1})} U_x$$

$\therefore$ $B_{1/2^n}(\ ) \supseteq V(p)$, an open set in R.

Now as, for each n, $U_n = U^*_n$, we have a decreasing chain of compact sets with intersection Δ. Then, by an elementary result, any open set V about Δ traps U_{n_0}, hence all the others for $n \geq n_0$.

Hence U_n° traps U_{m_n}, and all others after, for some m_n. Now this means that $D_{m_n} \subseteq U_n^\circ$, and so

$$B_{1/2^{m_n}} = \pi_1 [D_{m_n} \cap (R \times \{p\})]$$

$$\subseteq \pi_1 [U_n^\circ \cap (S \times \{p\})].$$

We must show that, for any $V(p)$, we can find a

$$U_n \ni \pi_1 [U_n^\circ \cap (S \times \{p\})] \subseteq V(p).$$

Now $\bigcap_{n=1}^{\infty} (U_n = U_n^* \cap S \times \{p\}) = \{(p,p)\}$, and thus,

$\exists\ n' \ni U_n^* \cap (S \times \{p\}) \subseteq V(p) \times V(p)$ for all $n \geq n'$, again by the entrapment theorem.

$\therefore \quad \pi_1 [U_{n'}^\circ \cap (S \times \{p\})] \subseteq V(p)$ and so we are done.

Lemma 2.29. Suppose $(R, +, \cdot)$ is a compact semiring, and that $\{U'_n : n = 1,2,3,\ldots\}$ is a sequence of subsets of $R \times R$ such that $\Delta \subseteq U_n'^\circ$, and $U'_{n+1} \subseteq U'_n$ for each n. Then $\exists$ a sequence $\{U_n : n = 0,1,2,\ldots\}$ of subsets of $R \times R$, with $U_0 = R \times R$, and satisfying, for each $n \geq 1$,

(i) $\Delta \subseteq U_n^\circ \subseteq U_n = U_n^*$

(ii) $U_n \circ U_n \circ U_n \subseteq U_{n-1}$

(iii) $U_n^{-1} = U_n$

(iv) $\Delta U_n \cup U_n \Delta \cup (\Delta + U_n) \cup (U_n + \Delta) \subseteq U_n$

(v) $U_n U_n \cup (U_n + U_n) \subseteq U_{n-1}$

(vi) $U_n \subseteq U^\circ_{n-1} \cap U_n$.

Proof. Define $U_0 = R \times R$ and $U_1 = U_1'^\circ \cap (U_1'^\circ)^{-1}$. Then clearly U_1 satisfies properties (i) - (vi). We assume that our set of $\{U_n\}$ is chosen up to $m = n-1$. We now construct U_n. Our method is first to construct a series of open sets V_1 through V_5. Let $\Delta \subseteq V_1 = U^\circ_{n-1} \cap U_n'^\circ$. Then, by Wallace's Theorem, $\exists\ \tilde{V}_2 \supseteq \Delta \ni \tilde{V}_2 + \tilde{V}_2 \cup \tilde{V}_2 \tilde{V}_2 \subseteq V_1$. By the full normality of our space, there is a $\tilde{\tilde{V}}_2 \ni \Delta \subseteq \tilde{\tilde{V}}_2 \subseteq \tilde{\tilde{V}}_2 \circ \tilde{\tilde{V}}_2 \circ \tilde{\tilde{V}}_2 \subseteq V_1$. $V_2 = \tilde{V}_2 \cap \tilde{\tilde{V}}_2$ has both of these properties. Then, again by repeated use of Wallace's Theorem and normality, we get a $\Delta \subseteq V_3$ such that $(V_3 + V_3) \cup (V_3 V_3) \subseteq V_2$, and $V_3^* \subseteq V_2$. Then again we get a $\Delta \subseteq V_a \ni (V_4 V_4 V_4) \cup (V_4 + V_4 + V_4) \cup (V_4 V_4) \cup (V_4 + V_4) \cup V_4 \subseteq V_3$. We next set $V_5 = V_4 \cap V_4^{-1}$, and finally, we set $V_6 = P_\Delta(V_5)$ (see Theorem 2.18). Thus $P_\Delta(V_5) \subseteq V_3$ and we define $U_n = V_6^*$.

Now we check the properties.

(i) $U_n^* = (V_6^*)^* = V_6^* = U_n \supseteq U_n^\circ \supseteq V_6^\circ \supseteq V_5 \supseteq \Delta$

(ii) $U_n \circ U_n \circ U_n = V_6^* \circ V_6^* \circ V_6^* \subseteq V_2 \circ V_2 \circ V_2 \subseteq V_1 \subseteq U^\circ_{n-1} \subseteq U_{n-1}$

(iii) $(P_\Delta(V_5))^{-1} = \bigcup_{i,j,k,\ell} (\Delta_i + \Delta_j V_5 \Delta_k + \Delta_\ell)^{-1}$

If $(x,y) \in (\Delta_i + \Delta_j V_5 \Delta_k + \Delta_\ell)$, $(y,x) = (p,p) + (q,q)(s,t)(r,r) + (z,z)$

$$= (p + qsr + z,\ p + qtr + z)$$

then $$(x,y) = (p + qtr + z,\ p + qsr + z).$$

But since $(s,t) \in V_5$, $\therefore$ $(t,s) \in V_5$, and thus $(x,y) \in \Delta_i + \Delta_j V_5 \Delta_k + \Delta_\ell$. We have only indicated one case, but even when some terms are missing, still

$$(\Delta_i + \Delta_j V_5 \Delta_k + \Delta_\ell)^{-1} \subseteq \Delta_i + \Delta_j V_5 \Delta_k + \Delta_\ell,$$

and so they are equal, hence $(P_\Delta(V_5))^{-1} = P_\Delta(V_5)$. Then, by continuity of $(\cdot)^{-1}$, we get that

$$U_n^{-1} = (V_6^*)^{-1} = (V_6^{-1})^* = V_6^* = U_n.$$

(iv) As remarked in Theorem 2.18, V_6 is a double Δ-ideal. Then, as the closure of a Δ-ideal is a Δ-ideal, we are done.

(v)
$$\begin{aligned}(U_n U_n) \cup (U_n + U_n) &\subseteq V_2 V_2 \cup (V_2 + V_2) \\ &\subseteq V_1 \subseteq U_{n-1}^\circ \subseteq U_{n-1}.\end{aligned}$$

(vi) $U_n \subseteq V_2 \subseteq V_1 \subseteq U_{n-1}^\circ \cap U_n'^\circ \subseteq U_{n-1}^\circ \cap U_n'$.

We have completed our induction.

Lemma 2.30. Let ρ be a closed congruence on the compact semiring $(R, +, \cdot)$. Let θ be the quotient map, and suppose that U is open in R. Then the set $V = \bigcup \{A : A \text{ is a } \rho\text{-class and } A \subseteq U\}$ is an open set in R, and so $\theta(V)$ is open in R/ρ.

Proof. Omitted. We discuss such matters in more detail in Section 10 of this chapter.

We come now to our main theorem for this section.

Theorem 2.31. Let $(R, +, \cdot)$ be a compact semiring. Then:

(i) If R is metrizable, R admits a subinvariant metric.

(ii) R is the inverse (projective) limit of an inverse system of compact metric semirings.

Proof. (i) For the given metric d, set

$$U'_n = \{(x,y)\colon d(x,y) < 1/n\}$$

$$= \bigcup_{x_0 \in R} \{(x_0,y)\colon d(x_0,y) < 1/n,\ x = x_0,\ y \in R\}$$

Then U'_n is open in $R \times R$, and clearly $U'_n \supseteq \Delta$. Moreover, $U'_{n+1} \subseteq U'_n$, and furthermore, $\bigcap_{n=1}^{\infty} U_n = \Delta$, so then, by Lemma 2.29, there is a sequence satisfying the requirements of Lemma 2.28, and so R admits a subinvariant metric.

(ii) Let

$$\mathcal{U}' = \{\{U'_n\}_{n=1}^{\infty}\colon U'_n \subseteq R \times R,\ \Delta \subseteq U'^{\circ}_n,\ U'_{n+1} \subseteq U'_n\}.$$

$\mathcal{U}' \neq \phi$, because we have $\{U'_n\}_{n=1}^{\infty}$, where each $U'_n = R \times R$. Now, by Lemma 2.29, to each member of $\mathcal{U}'$ there corresponds a sequence $\{U_n\}_{n=1}^{\infty}$ with the following properties.

(i) $\Delta \subseteq U_n^{\circ} \subseteq U_n = U_n^{*}$

(ii) $U_n \circ U_n \circ U_n \subseteq U_{n-1}$

(iii) $U_n^{-1} = U_n$

(iv) U_n is a double Δ-ideal

(v) $(U_n U_n) \cup (U_n + U_n) \subseteq U_{n-1}$

(vi)' $U_n \subseteq U^\circ_{n-1}$.

We first remark that $R = \bigcap_{n=1}^{\infty} U_n$ is a closed semiring congruence. The only thing which may not be clear is the transitive law. Let (x,t) and $(t,z) \in R$. Then $(x,z) \in U_n \circ U_n$ for each n, and $(z,z) \in U_n$ for each n, $\therefore$ $(x,z) \in U_n \circ U_n \circ U_n \subseteq U_{n-1}$ for each n, and $\therefore$ $(x,z) \in R$. We obtain, for every possible sequence in R x R with properties (i) - (vi)', the corresponding R, and then we form the set $K = \{R\colon R$ is formed as just described$\}$.

If $\{U_n\}$ and $\{V_n\}$ are two such sequences, form $W_n = U_n \cap V_n$, which clearly also satisfies (i) - (vi)'. Then the corresponding R will be included in the intersection of the other two. Moreover, $\bigcap K = \Delta$, because if not, we choose $x \neq y$, $(x,y) \in R$, for each $R \in K$. Then $W = R \times R \setminus \{(x,y)\}$ is an open set containing Δ. Then form $\{U'_n\} \in \mathcal{U}'$, by $U'_n = W$ for each $n \geq 1$. Then, using Lemma 2.29, we can get a sequence $\{U_n\}$, such that $(x,y) \notin R = \bigcap_{n=1}^{\infty} U_n$. Thus, under containment, K is a directed set and $\bigcap K = \Delta$. So, by our Corollary 2.26, R is iseomorphic to $\varprojlim \{R/R\colon R \in K\}$. It remains to show that each R/R is metric, where $R = \bigcap_{n=1}^{\infty} U_n$. Consider $\{\tilde{U}_n\}$ where $\tilde{U}_n = (\theta \times \theta)\, U_n$ $(\theta\colon R \to R/R)$. To show that $\bigcap \tilde{U}_n = \tilde{\Delta}$, we refer the reader to similar arguments in Section 10. The remainder of the points to be satisfied for Lemma 2.28 are easily verified, except possibly that $\tilde{\Delta} \subseteq \tilde{U}^\circ_n$. We give the necessary argument for that part.

Let $(x,x) \in \Delta$. Then $\theta^{-1}([x]) \times \theta^{-1}([x]) \subseteq U_n$, so, by Wallace's Theorem, $\exists V\, (\theta^{-1}([x])) \times V\, (\theta^{-1}([x])) \subseteq U_n$. By Lemma 2.30, if $T = \cup\{A\colon A$ is an $\mathcal{R}$ class and $A \subseteq V\}$, then T is open in R, and $T \subseteq V$. Moreover, $\theta(T)$ is open in $R/\mathcal{R}$, and also,

$$\begin{aligned}\tilde{U}_n = (\theta \times \theta)\, U_n &\supseteq \theta(V(\theta^{-1}[x])) \times \theta(V(\theta^{-1}([x]))) \\ &\supseteq \theta(T) \times \theta(T) \ni ([x], [x]),\end{aligned}$$

and $\theta(T) \times \theta(T)$ is open in $R/\mathcal{R} \times R/\mathcal{R}$, so that $\tilde{U}_n^{\circ} \supseteq \tilde{\Delta}$.

<u>Theorem 2.32</u>. Let $(R, +, \cdot)$ be a compact semiring, and let W be a compact open subset of $R \times R$ such that $W \supseteq \Delta_R$. Then there exists a clopen (open and closed) subset D, such that $W \supseteq D \supseteq \Delta_R$, and D is a semiring congruence on R.

<u>Proof</u>. By Theorem 2.22, there is a morphism $\theta\colon R \to T$, of R onto a totally disconnected semigroup T, such that θ is the monotone factor of the monotone-light factorization of the constant map. Then $\theta \times \theta\colon R \times R \to T \times T$, by the properties of connected sets, is the monotone map of the monotone-light factorization of the constant map on $R \times R$. Since $W \subseteq R \times R$ is compact and open, it is a union of components of $R \times R$. Hence, letting $B = (\theta \times \theta)\,(W)$, we get that B is a compact open subset of $T \times T$ which contains the diagonal. Also, $W = (\theta \times \theta)^{-1}\,(B)$, so we need only find a clopen set D', of $T \times T$, such that $B \supseteq D' \supseteq \Delta_T$, and then take $D = (\theta \times \theta)^{-1}\,(D')$. So we assume now that $(R, +, \cdot)$ is totally disconnected.

By Numakura [19], the diagonal of $R \times R$ has a basis of compact open sets which are equivalence relations on R. Thus we assume, without loss of generality, that W is an equivalence relation on R. Now, by our Lemma 2.19, taking as our semiring, $R \times R$, and as our closed subsemigroup, Δ, there is an open subset of $R \times R$, say V, with the property that $V \subseteq W$, and $\Delta V \cup V\Delta \cup (\Delta + V) \cup (V + \Delta) \subseteq V$.

<u>Note</u>: Although it may seem that we have merely stated for two operations what might well be true for any number of operations, this does <u>not</u> appear to be the case, because the lemma that we used relied heavily on the distributivity of $(\cdot)$ over $(+)$.

Then the smallest closed equivalence relation D, containing V, is a semiring congruence, by Corollary 2.24, moreover, we show that, in the present case, $V = D$. The argument is merely a trivial change in that of Numakura [19]. First, as he notes, $V^{-1} = \{(x,y): (y,x) \in V\}$ is also a double Δ-ideal contained in W, so must, by our lemma, be contained in V, since V is maximal. Consider then, the transitive closure of V.

$$B = \bigcup_{n=1}^{\infty} (V \circ V \circ \ldots \circ V)_{(n \text{ factors})} \subseteq W.$$

If $(p,q) \in \Delta\,(V \circ V \circ \ldots \circ V)$ $(= \Delta\, V^{(n)})$, then $\exists\ (q_0,q_1), (q_1,q_2),\ldots,(q_{n-1},q_n) \in V$, and $(x,x) \in \Delta$, such that $(p,q) = (xq_0, xq_n)$. But $(xq_0,xq_1), (xq_1,xq_2),\ldots,(xq_{n-1},xq_n) \in V$, since V is a double Δ-ideal, and therefore $(p,q) \in V^{(n)}$. Thus $\Delta\, V^{(n)} \subseteq V^{(n)}$, and so $\Delta B \subseteq B$. Similarly,

$B\Delta \cup (\Delta + B) \cup (B + \Delta) \subseteq B$, and so B is a double Δ-ideal of $R \times R$, which is contained in W, and then $B \subseteq V$. So V is an equivalence, hence V is an open semiring congruence, hence a clopen one.

Corollary 2.33. Every compact, totally disconnected semiring, is a strict projective limit of finite discrete semirings.

Proof. By the Theorem 2.32, and its Proof, there is a collection $\mathcal{D}$ of clopen congruences D, which have the diagonal Δ for their intersection (hence they are directed down, by containment). Then, by Theorem 2.25, R is a strict projective limit of semirings R/D, which are compact, and discrete, because D is clopen.

Theorem 2.34. Let $(R, +, \cdot)$ be a compact semiring with double absorber z, (i.e., $x + z = z + x = xz = zx = z$, for each $x \in R$), and let $\mathcal{J}$ be a collection of closed neighborhoods of z which are double ideals, and such that $\mathcal{J}$ is a basis for the neighborhoods of z. (Such a collection must exist by Lemma 2.19.) Then R is a projective limit of semirings R/I, $I \in \mathcal{J}$.

Proof. Inspect Lemma 2.19, using $T = R$, and then use Theorem 2.25.

7. A Remark Concerning Certain Very Special Compact Semirings--Namely, Self-Distributive Compact Semigroups

Definition 2.35. A semigroup $(S,\cdot)$, is self-distributive, if

(i) $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$ and

(ii) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$

are satisfied. Clearly, by letting $\circ$ stand for both $(\cdot)$, and $(+)$, we have a compact semiring. We mention one result.

Theorem 2.36. In a self-distributive compact semigroup all $\mathcal{H}$-classes are singletons.

Proof. The Schutzenberger group of the $\mathcal{H}$-class is homeomorphic to it. Then this must be a self-distributive group, and so it must be a singleton.

8. General Comments About Compact Semimodules Over Compact Semirings

We note that there is a trivial action of any compact semiring $(R, +, \cdot)$, on any idempotent semigroup $(M,+)$, by defining $(r,x) \mapsto x$. We are more interested to restrict consideration to cases where $r_1 \neq r_2$, implies $\exists\, x \in X \ni r_1 x \neq r_2 x$.

Definition 2.37 (R-M congruence). If E is a semigroup congruence on $(M,+)$, with the additional property that for each $r \in R$, and for each $(x,y) \in E$, we get $(rx,ry) \in E$, then we call E an R-M congruence on M.

Definition 2.38 (R-M morphism). $\theta: M_1 \to M_2$, a continuous homomorphism, will be called an R-M morphism, if the following diagram commutes:

$$\begin{array}{ccc} R \times M_1 & \to & M_1 \\ i \times \theta \downarrow & & \downarrow \theta \\ R \times M_2 & \to & M_2 \end{array} .$$

The next theorem follows by similar arguments to that in action theory.

Theorem 2.39. If $R \times M \xrightarrow{f} M$, where f is the semiring action of R on M (both compact), then for E, any closed R-M congruence on M, $\exists$ one, and only one, continuous function α, $\ni$ the following diagram commutes:

$$\begin{array}{ccc} R \times M & \xrightarrow{f} & M \\ i \times \theta \downarrow & & \downarrow \theta \\ R \times M/E & \xrightarrow{\alpha} & M/E \end{array} .$$

Moreover, M/E is a semimodule over R, via α.

Proof. Omitted.

So, we can restrict our attention, by factoring out R-M congruences in this way. Similarly, for any closed semiring congruence $\mathcal{R}$, on R, we get a canonical action of $R/\mathcal{R} \times M \to M$, giving that M is a semimodule over $R/\mathcal{R}$. More generally, if $\mathcal{R}\,\mathcal{E} \subseteq \mathcal{E}$ (when we consider $(R \times R) \times (M \times M) \to M \times M$, the product semimodule), then it is also true that $R/\mathcal{R} \times M/\mathcal{E} \to M/\mathcal{E}$ is uniquely defined as a semimodule action of $R/\mathcal{R}$ on $M/\mathcal{E}$. Examples of these congruences are:

$\mathcal{R} = \{(r_1,r_2) \in R \times R$: for each $m \in M$, $\quad r_1m = r_2m\}$

$\mathcal{E} = \{(m_1,m_2) \quad M \times M$: for each $r \in R$, $\quad rm_1 = rm_2\}$, where

we get that $\mathcal{R}\,\mathcal{E} = \Delta$.

This type of approach enables one to eliminate certain types of trivial cases while carrying out investigations, or, in other words, enables a type of representation theory. However, we will not pursue this further here.

Carrying what has been done in semigroup actions (and in this dissertation) in regard to minimality, and maximality of "ideal," a step further, we state (without proof) the following:

Theorem 2.40. If $(M,+)$ is a compact semimodule over the compact semiring $(R, +, \cdot)$, then M contains a unique closed subset B, which is minimal with respect to the property that $(M + B) \cup (B + M) \cup R\,B \subseteq B$. We call B the minimum R-subvariant ideal of M. Moreover, if $B \neq M$, then B is contained in a (not

necessarily unique) maximal proper R-subvariant ideal of M, say J, and J is open in M.

Of course $R \times B \to B$ is a semimodule action of R on B. We may now profitably state

Theorem 2.41. Let (M,+) be a compact abelian semimodule over the compact semiring $(R, +, \cdot)$. Suppose further that $(R,\cdot)$ is a homogroup, and that M has no proper closed subvariant additive ideal. Then M contains an element z with the following properties:

(i) $R\, z = z$,

(ii) z is an additive identity for M,

(iii) $K\,[\cdot]\, M = z$.

Proof. By a result of Pearson [21], $(K\,[\cdot], +)$ is a rectangular band. Choose an $x \in M$. Then $K\,[\cdot]\, x$, being a morphic image of $(K\,[\cdot], +)$, is also a rectangular band. However, as (M,+) is abelian, we get that $K\,[\cdot]\, x = z_x$, a singleton. Now $R\, z_x = R\,(K\,[\cdot]\, x) \subseteq K\,[\cdot]\, x = z_x$, and so $R\, z_x = z_x$. We show below that z_x is independent of x, and will be our z. Since $R\,(z_x + M) \subseteq R\, z_x + R\, M \subseteq z_x + M$, therefore $z_x + M$ is a subvariant ideal of M, and thus, by hypothesis, $z_x + M = M$. We already have $z_x + z_x = z_x$ and so z_x is an additive identity for M. Our argument so far is independent of x, and thus, $z_x = z_x + z_{x'} = z_{x'}$. Now if we put $z_x = z$, we have $K\,[\cdot]\, M = z$.

Consider the following abstraction.

<u>Theorem 2.42</u>. Let $(M_1,+)$, $(M_2,+)$, and $(M_3,+)$, be compact semigroups, together with a function $M_1 \times M_2 \to M_3$, denoted by juxtaposition, and with the properties:

(i) $(a + b)c = ab + ac$, for all $a, b \in M_1$, $c \in M_2$,

(ii) $a(b + c) = ab + ac$ for all $a \in M_1$, $b, c \in M_2$.

Then $M_1 M_2 \cap K_3 \neq \phi$, implies $K_1 K_2 \subseteq K_3$.

<u>Proof</u>. Let $x_1 x_2 \in K_3$, where $x_1 \in M_1$, and $x_2 \in M_2$. Then $M_1 x_2 \cap K_3 \neq \phi$, and so $T = M_1 x_2 \cap K_3$ is the minimal ideal of $M_1 x_2$, thus giving $K_1 x_2 = T \subseteq K_3$. Similarly, $x_1 K_2 \subseteq K_3$. Now letting $p \in K_1$, as $p x_2 \in K_1 x_2 \subseteq K_3$, by repeating our argument above we obtain $p K_2 \subseteq K_3$, and so $K_1 K_2 \subseteq K_3$.

We note that the situation arises when $(M_3,+)$ is a compact semimodule $(M,+)$ over a compact semiring $(R, +, \cdot)$. Then letting $(M_1,+)$ be $(R,+)$, and $(M_2,+)$ be $(M,+)$, and letting the anonymous function be the semiring action on M, we obtain

<u>Corollary 2.43</u>. If $(M,+)$ is a compact semimodule over the compact semiring $(R, +, \cdot)$, then $R M \cap K_M \neq \phi$ implies that K_R [+] $K_M \subseteq K_M$.

We note also, that this contains the following result of Pearson [22], which he obtained by a pointwise argument.

Corollary 2.44. If $(R, +, \cdot)$ is a compact semiring, then $R^2 \cap K[+] \neq \phi$ implies that $K[+]$ is a subsemiring of R.

We record certain generalizations of work of Selden [26], due to a result of Wallace [35].

Theorem 2.45. If $(M,+)$ is a compact semimodule over the compact semiring $(R, +, \cdot)$, and if $m \in Rm$ for each $m \in M$, then each subgroup of M is totally disconnected.

Corollary 2.46. If $(M,+)$ satisfies the condition of the theorem, and as well is connected, then the minimal ideal K of M consists of idempotents.

Proof. Each subgroup of K is connected, and totally disconnected, hence a singleton.

Corollary 2.47. If $(M,+)$ satisfies the conditions of the theorem, and as well is connected, normal, and has an additive identity 0, then, provided M is not a singleton, it contains an arc from its singleton minimal ideal to 0. If M is also metric, then it is arcwise connected.

Proof. This follows using that the subgroups are totally disconnected. (See Selden [25].)

9. Nearrings, Group Morphisms, and Semimodules Over Semirings

This section was directly inspired by work of Wallace [35], and, indeed, any depth in the work is due to him; the author's contribution being merely to observe that certain conditions in Wallace's theorems could be relaxed considerably, thus widening the applicability. We begin by stating and proving the most general statement that we can make, then we go on to indicate how other results follow as corollaries.

Theorem 2.48. Let G be a compact group, and suppose T_1 and T_2 are Hausdorff spaces such that T_2 is compact. We further suppose that there is a continuous function $T_1 \times T_2 \to G$, such that, for each $s \in T_1$, sT_2 is a (compact) group. Then, if $q \in T_1$, and $qT_2 = 1$, it follows that $CT_2 = 1$ (where C is the component of T_1 to which q belongs). (There is an obvious dual.)

Proof. We first state two results involving Lie groups. (See Montgomery-Zippin [17].)

(1) A compact group is a Lie group, if, and only if, there is some open set about the neutral element which properly contains no closed subgroup other than the trivial one.

(2) If G is a compact group, and if U is any open set about the neutral element, then there is a morphism f (continuous!), of G onto a Lie group with ker $f \subseteq U$.

We continue with the proof of the theorem. Let f be any morphism of G onto a Lie group L, and let $I(f) = \{t \in T_1: f(tT_2) = 1 \in L\}$. I(f) is not empty, since we suppose $qT_2 = 1 \in G$, and then $f(qT_2) = f(1) = 1 \in L$, since f is a morphism. Let $t_0 \in I(f)$. Then $f(t_0T_2) = 1$. For V an open set about $1 \in L$ such that V contains no nontrivial closed subgroup of L (possible by (1)), we put $W = f^{-1}(V) \supseteq t_0 T_2$. Then, by the compactness of T_2, $\exists W'$, open in T_1, such that $t_0 \in W'$, and $W \supseteq W'T_2$. Now, if $t' \in W'$, then $f(t'T_2) \subseteq V$. But $t'T_2$ is a closed subgroup of G, and so $f(t'T_2)$ is a closed subgroup of L. By our manner of choosing V, we have that $f(t'T_2) = 1$ giving $W' \subseteq I(f)$ and, since t_0 was arbitrary, we have that I(f) is open.

Also,
$$\begin{aligned} f(I(f)^* T_2) &\subseteq f((I(f)\,T_2)^*) \\ &\subseteq [f(I(f)\,T_2)]^* \\ &\subseteq \{1\}^* \\ &= 1. \end{aligned}$$

Thus I(f) is closed, and open, as well as nonempty, and so $C \subseteq I(f)$. Suppose that $x \in C$, and $y \in T_2$, with $xy = 1 \in G$. Then $G \setminus \{xy\}$ is an open set in G to which 1 belongs. However, by (2) above, we may choose our f, and L, such that $f^{-1}(1) \subset G \setminus \{xy\}$. But then $f(xy) \neq 1 \in L$, and this is a contradiction because $f(xy) \in f(I(f)\,T_2) = \{1\}$.

Corollary 2.49. Let $(R, +, \cdot)$ be a compact nearring (Definition 2.3). Then, if we denote the connected component of R which contains the neutral element 0 of R by C, we have $CR = 0$.

Proof. Take T_1, T_2 and G all as R. However, we write additively, so that the 1 of the theorem now appears as 0. The map $R \times R \to R$ is the multiplication of the nearring. 0 plays the role of q. For each $s \in R$, sR is seen to be a compact subsemigroup of (R,+), because $sr_1 + sr_2 = s(r_1 + r_2) = sr$. Then of course, as mentioned earlier, in such a case sR must be a subgroup of (R,+), and we get immediately that CR = 0.

The above result generalizes work by Beidleman and Cox [2]. The following result of Wallace [35], which gave rise to our theorem, clearly follows as a corollary to our result and its dual.

Theorem 2.50. If (M,+) is a compact additive group which is an R-semimodule over the compact semiring (R, +, $\cdot$), with (R,+) a group, then in both (R,+), and (M,+), denoting the neutral element by 0, and the connected component of R, respectively M which contains 0, as C, respectively D, we have that

(i) $R\,D = 0$

(ii) $C\,M = 0$.

Now we can gain a little more by looking at the theorem a little differently. Let us denote by T the set of algebraic homomorphisms from a compact group G_1 into a compact group G_2 such that the homomorphisms are continuous maps. Let us consider two cases. One case is where G_2 is abelian and the other is where

$G_1 = G_2 = G$ is not abelian, but is connected. We consider T as a topological space with the compact-open topology induced by the groups in the standard way. Then $T \times G_1 \to G_2$, by evaluation, is a continuous map.

Corollary 2.51. In case G_2 is abelian T is totally disconnected.

Proof. The trivial homomorphism t, which takes all of G_1 onto the neutral element of G_2, has the property of q in Theorem 2.48 if we identify T_1 with T, T_2 with G_1, G with G_2, and the 0 of G_2 with the 1 of G. Certainly sG_1 is a compact subgroup of G_2. Thus, applying the proposition, we obtain that for each t' in the component of T containing t, we get $t'G_1 = 0$. However, this means that $t' = t$. Thus the component of T containing t is just $\{t\}$. I am indebted to Professor Sigmon for pointing out that since G_2 is abelian, we may construct a topological group on T using pointwise defined addition, and the compact-open topology. It then follows that T is totally disconnected.

Corollary 2.52. In case $G_1 = G_2 = G$ is not abelian, but is connected, then T is nonhomogeneous.

Proof. Once again, taking $T_1 = T$, $T_2 = G$, and $G = G$ in Theorem 2.48, we have that the component of T containing the trivial homomorphism t, is $\{t\}$. Now, however, by mapping G into T in the following way: $g \leftrightarrow f_g : G \to G$, where $f_g(x) = g^{-1} x g$, we produce

a continuous image of G in T which contains more than one element, and so we have a connected subset of T with more than one element, which means that T is nonhomogeneous.

The next two propositions are generalizations of results of Wallace [35], and Selden [26].

Theorem 2.53. If $(M,+)$ is a compact semimodule over the compact semiring $(R, +, \cdot)$, and T is a divisible subsemigroup of M with an identity in R (i.e., an element $a \in R$, such that $a\,T = T$), then T consists entirely of idempotents.

Proof. Without loss of generality $T = T^*$. Then, due to the divisibility of T,

$$(a + a + \dots + a)\,T = a\,T + a\,T + \dots + a\,T \quad \text{(n terms added)}$$
$$= T + T + \dots + T$$
$$= T \quad \text{(again using divisibility).}$$

Then $\{x \in R: x\,T = T\}$ is closed, and contains $a + a + \dots + a$, for any number of a's, so contains an element f such that

$$f = f + f \quad (f \text{ is the idempotent of } \Gamma[+](a)).$$

But, if $t \in T$, then $\exists\, t' \in T$ such that $t = f\,t'$.

Then $$t + t = ft' + ft'$$
$$= (f + f)\,t'$$
$$= ft'$$
$$= t\ .$$

Theorem 2.54. If $(M,+)$ is a compact semimodule over the compact semiring $(R, +, \cdot)$, such that for each $x \in M$, $x \in R\,x$, then each abelian divisible subsemigroup T of M consists entirely of idempotents.

Proof. Without loss of generality $T = T^*$. Choose $t \in T$, and let C be the component of T to which t belongs. Then by a result of John Hildebrandt [9], C is a compact divisible subsemigroup of T. Without loss of generality $\exists\, e^2 = e \in R$ such that $et = t$. Considering eC, $t \in e\,C$ and eC is connected, so, $eC \subseteq C$. Also, $eC + eC = e\,(C + C) \subseteq eC$. So eC is a compact subsemigroup of M. It is clear that eC is divisible, because C is divisible. Moreover, $e\,(eC) = eC$, and then, by our previous theorem, we have that eC consists entirely of idempotents. Since $t \in eC$, $t + t = t$. However, t was arbitrary in T, so we are done.

10. Product Structure in Special Types of Compact BATS

In what follows M will stand for either a semiring or a semigroup. Whenever topology is involved we mean the corresponding BAT. Congruence shall mean a semiring congruence, or, a semigroup congruence, according to the meaning given M.

We begin with two lemmas which are quite simple, but basic for what follows.

Lemma 2.55. If σ and ρ are congruences on M, such that $\rho \supseteq \sigma$, then denoting the canonical map from M to M/σ by θ, it follows that

$$\rho = (\theta \times \theta)^{-1} [(\theta \times \theta)(\rho)]$$
$$= \{(x,y) \in M \times M\colon (\theta(x), \theta(y)) \in (\theta \times \theta)(\rho)\}$$

Proof. We certainly have $\rho \subseteq (\theta \times \theta)^{-1} [(\theta \times \theta)(\rho)]$. Let $(x,y) \in (\theta \times \theta)^{-1} [(\theta \times \theta)(\rho)]$. Then $\exists\ (a,b) \in \rho \ni (\theta(x), \theta(y)) = (\theta(a), \theta(b))$, and so we have $\theta(x) = \phi(a)$, and $\theta(y) = \phi(b)$. That is, (x,a) and (y,b) are in σ. Since $\sigma \subseteq \rho$, we have (x,a) and (y,b) in ρ. However, we already have $(a,b) \in \rho$, and, by symmetry, $(b,y) \in \rho$. Then, by transitivity, we have $(x,y) \in \rho$.

Lemma 2.56. If σ and ρ are congruences on M such that $\rho \supseteq \sigma$, and ρ is maximal, then, denoting the canonical map from M to M/σ by θ, we have that

$$(\theta \times \theta)(\rho) = \{(\theta(a), \theta(b))\colon (a,b) \in \rho\}$$

is a maximal congruence on M/σ.

Proof. It is trivial that $(\theta \times \theta)(\rho)$ is a congruence on M/σ. Also, it is seen to be proper, for, by Lemma 2.55,

$$\rho = (\theta \times \theta)^{-1} [(\theta \times \theta)(\rho)].$$

If it is assumed that $(\theta \times \theta)(\rho)$ is properly contained in a proper congruence τ, then $(\theta \times \theta)^{-1}(\tau)$ is seen to be a proper congruence

on M which properly contains ρ, and this is a contradiction. Thus $(\theta \times \theta)(\rho)$ is a maximal congruence on M/σ.

From now on we are interested in the following property which holds in rings and groups.

Property X. Whenever J is a maximal congruence on M, and B is a proper congruence on M such that $B \nsubseteq J$, then every J class meets every B class.

Theorem 2.57. If M has property X, so does any morphic image of M.

Proof. Suppose we have $\theta\colon M \twoheadrightarrow M'$, and M has property X. Then, if J' is a maximal congruence on M', and if B', is a proper congruence on M' such that $B' \nsubseteq J'$, we will consider

$$J = (\theta \times \theta)^{-1} (J') \quad \text{and}$$

$$B = (\theta \times \theta)^{-1} (B') \ .$$

J is a maximal congruence on M, for, if not, we obtain a contradiction using Lemma 2.56. Also, B is a proper congruence on M. Still again $B \nsubseteq J$, for, if $B \subseteq J$, then $B' \subseteq J'$, which is not so. Thus, by property X in M, we can say that each J class meets each B class. However, then the image of the nonvoid intersection will be contained in the intersection of the images of the respective J and B classes. Moreover, by our definition, each J class is the inverse image under

θ of a J' class, and similarly, each B class is the inverse image under θ of a B' class, so that we have property X for M'.

<u>Definition 2.58</u>. M is said to be <u>congruence free</u> if the only congruences on it are $M \times M$ and Δ.

<u>Theorem 2.59</u>. Let M be compact, and have property X. Let $J_1, \ldots, J_n$ be closed, maximal congruences on M, such that $J_1 \cap \ldots \cap J_t \nsubseteq J_{t+1}$, for each $t = 1, \ldots, n-1$. Then, with the canonical topologies, and the canonical map, we have

$$M/(J_1 \cap \ldots \cap J_n) \overset{t}{\cong} M/J_1 \times M/J_2 \times \ldots \times M/J_n .$$

Each of the M/J_i is a congruence free compact BAT of the same type as M, and, if J_i is open, then M/J_i is finite and discrete.

<u>Proof</u>. By Sierpinski's Lemma, the following diagram may be completed as shown:

$$\begin{tikzcd} M \arrow[r, two heads, "\theta_i"] \arrow[dr, two heads, "\psi"'] & M/J_i \\ & M/(J_1 \cap \ldots \cap J_n) \arrow[u, dashed, two heads, "\xi_i"'] \end{tikzcd}$$

by the canonical map ξ_i, and this map is a continuous morphism. As is well known, each of the spaces is compact Hausdorff, and the multiplication (or addition) is continuous in the canonical quotient topology. When J_i is open, then, since M is compact, there is only a finite number of congruence classes, so that M/J_i is finite, and, hence, has the discrete topology. Lemmas 2.55, and 2.56 above, may be used to show quickly that M/J_i is always congruence free.

Now consider:

$$M/(J_1 \cap \ldots \cap J_n) \xrightarrow{\prod_{i=1,\ldots,n} \xi_i} M/J_1 \times M/J_2 \times \ldots \times M/J_n \ .$$

Since our product has the canonical product topology induced by the M/J_i, and each ξ_i is continuous, it is well known that $\prod_{i=1,\ldots,n} \xi_i$ is continuous. Also, it is obviously a morphism. That it is $1 - 1$ is also obvious. The only thing to show, is that the map is onto. We prove this by finite induction. First, however, we note that our original theorem is trivially true if $n = 1$, so that we all the time consider $n \geq 2$. The first step in our induction then, is to demonstrate that our map is onto in the case $n = 2$. Certainly $J_1 \nsubseteq J_2$, so that, by property X, each J_2 class meets each J_1 class. This is the same as saying our map is onto. By induction assumption, since the set $J_1,\ldots,J_{n-1}$ has the property $J_1 \cap \ldots \cap J_t \nsubseteq J_{t+1}$ for each $t = 1,\ldots,n-2$ (which follows from our original assumption on the set $J_1,\ldots,J_n$), we may assume that

$$M/(J_1 \cap \ldots \cap J_{n-1}) \cong M/J_1 \times \ldots \times M/J_{n-1} .$$

Now $M/(J_1 \cap \ldots \cap J_n) = M/((J_1 \cap \ldots \cap J_{n-1}) \cap J_n)$,

and since $J_1 \cap \ldots \cap J_{n-1} \nsubseteq J_n$ (from our hypotheses), then, in the same way as case $n = 2$ above, we see that

$$M/(J_1 \cap \ldots \cap J_n) \cong M/(J_1 \cap \ldots \cap J_{n-1}) \times M/J_n .$$

So altogether we may say

$$M/(J_1 \cap \ldots \cap J_n) \cong M/J_1 \times M/J_2 \times \ldots \times M/J_{n-1} \times M/J_n ,$$

the map being the canonical one. Our argument here, being purely algebraic, has shown that our finite induction to prove onto is complete. So, our whole argument is complete, and the theorem is proved.

We will return to this result later when it is used in the proof of Theorem 2.67.

Definition 2.60. Let M be compact, and let $T(M)$ be the set of, open and maximal, congruences on M, if there are any; otherwise, $T(M)$ is the singleton set containing $M \times M$ as its only element. We define the radical of M, by

$$R(M) = \cap \; T(M) .$$

If $R(M) = \Delta$, we say that M is semisimple. We note that $R(M)$ is closed, and so $M/R(M)$ is again a compact BAT. We further note that, if we replace the word open above by closed, $T(M)$ by $\tilde{T}(M)$, and $R(M)$ by $R(M)$, our definition and remarks still make sense. If $\tilde{R}(M) = \Delta$ we say that M is weakly semisimple.

<u>Theorem 2.61</u>. If M is compact, then,

$$R(M/R(M)) = \Delta .$$

(The same theorem is true for $\tilde{R}(M)$ but the proof is easier.)

<u>Proof</u>. <u>Case 1</u>. $T(M) = \{M \times M\}$. Trivial.

<u>Case 2</u>. There is at least one, maximal and open, congruence on M. Then let $\rho \in T(M)$. Denoting as usual the canonical map from M to $M/R(M)$ by θ, the set $(\theta \times \theta)(\rho)$ is an, open and maximal, congruence on $M/R(M)$. This is because: (i) By Lemma 2.55 above, $(\theta \times \theta)(\rho)$ is a maximal congruence on $M/R(M)$ and: (ii) Each $(\theta \times \theta)(\rho)$ class, say A, has for its preimage a ρ class A', and $M \setminus A'$ is a closed set. Then, since M is compact, and θ is a closed map onto $M/R(M)$, we have that $[M/R(M)] \setminus A = \theta(M \setminus A')$, and so $[M/R(M)] \setminus A$ is closed, and so A is open. Now choose an arbitrary pair $(p,q) \in [M/R(M)] \times [M/R(M)]$ so that $p \neq q$. We have $p = \theta(a)$, $q = \theta(b)$ for some a and b in M, and also $(a,b) \notin R(M)$. Thus, there exists an, open and maximal, congruence, say J, on M such that $(a,b) \notin J$. Then $(\theta(a), \theta(b)) \notin (\theta \times \theta)(J)$ since, by Lemma 2.55, $J = (\theta \times \theta)^{-1} [(\theta \times \theta)(J)]$. However, as already noted, $(\theta \times \theta)(J)$ is

an, open and maximal, congruence on $[M/R(M)] \times [M/R(M)]$. So $(p,q) \notin R(M/R(M))$ and thus $R(M/R(M)) = \Delta$.

Theorem 2.62. Any congruence ρ, on a compact M, which is maximal in the set T (assumed nonempty) of, open proper, congruences on M, is a maximal congruence on M. That is, ρ is an, open and maximal, congruence on M.

Proof. Since ρ is open it is also closed. Since M is compact, M/ρ is finite and has the discrete topology. So then $M/\rho \times M/\rho$ is also finite, and also has the discrete topology. Suppose that $M \times M \supsetneq \tau \supsetneq \rho$, and τ is a congruence on M. Then, denoting the canonical map from M to M/ρ by θ, we have that $(\theta \times \theta)(\tau)$ is an open congruence on $M/\rho \times M/\rho$ because there, all congruences are open, since we have the discrete topology. Then, by Lemma 2.55, we have that $\tau = (\theta \times \theta)^{-1} [(\theta \times \theta)(\tau)]$, and τ is open because $\theta \times \theta$ is continuous, since θ is continuous. This contradicts the maximality of ρ in T, and thus, ρ is not only maximal in the set T, but is a maximal congruence on M.

Theorem 2.63. If M is compact, and has an, open and proper, congruence σ on it, then it has a congruence ρ on it which is both a maximal congruence, and an open congruence. (We note that ρ is of course also closed.)

Proof. By a simple application of Zorn's Lemma, there is at least one congruence on M, say ρ, which is maximal in the set of, open and proper, congruences on M, and contains σ. Then, by Theorem 2.62, ρ is also a maximal congruence on M.

Corollary 2.64. Any nonsingleton, compact, totally disconnected M has a congruence on it which is both a maximal, and an open congruence.

Proof. Let $(x,y) \in M \times M$, with $x \neq y$. Then $V = M \times M \setminus \{(x,y)\}$ is an open set in $M \times M$ which contains the diagonal Δ. Then either by Numakura [19] in the semigroup case, or by the proof of Theorem 2.32 for the semiring case, V contains an open congruence σ on M. Of course σ is proper, and Theorem 2.63 applies directly.

Theorem 2.65. Any nonsingleton, compact, totally disconnected M has a nontrivial semisimple homomorphic image, and a nontrivial weakly semisimple image.

Proof. This follows directly from Theorem 2.61 and the Corollary 2.64.

We now prove another fundamental lemma concerning congruences on M.

Lemma 2.66. If I is a set of congruences on M such that $\bigcap I = \Delta$, then I may be well ordered such that, for some subset I' of I with the well ordering induced from that on I, we will have the following two properties satisfied:

(i) $\bigcap I' = \Delta$.

(ii) If $T'_{\alpha_1},\ldots,T'_{\alpha_n}$ is any finite set from I' arranged in increasing order by the well order on I', then $(T_{\alpha_1} \cap \ldots \cap T_{\alpha_{n-1}}) \not\subseteq T_{\alpha_n}$.

Proof. Well order I, and index with an ordinal k. Choose a subset of I as follows:

If $k = 0$, $\quad I' = I = \{T_0\}$.

Otherwise, define a function $f: I \to \{0,1\}$ by the following procedure. Set $f(T_0) = 1$, and let α be any ordinal such that $k \geq \alpha > 0$. Then let $B_\alpha = \{\beta\colon \beta < \alpha \text{ and } f(T_\beta) = 1\}$ ($\neq \phi$ as $T_0 \in B$). If, for every finite subset of B, $\beta_1,\ldots,\beta_n$ (n a natural number) in increasing order, we have $T_{\beta_1} \cap \ldots \cap T_{\beta_n} \not\subseteq T_\alpha$, then we set $f(T_\alpha) = 1$, otherwise, we set $f(T_\alpha) = 0$.

We note that we do have a function. For, if not, choose the least member T_α of I for which our function is not defined. This is possible since I is well ordered. Certainly $\alpha > 0$, so that B_α exists and is not empty. Then we may choose $f(T_\alpha)$ unambiguously as above, which is a contradiction. So f is defined on all of I. A similar argument shows that our f is uniquely defined. Now define I'; by $I' = f^{-1}(1)$. We must show that I' has properties (i) and (ii).

We note first that the order induced on I' by I is a well order on I'.

(i) If $x \neq y$, and $(x,y) \in T_{\alpha'}$ for each $T_{\alpha'} \in I'$, then $\exists\, T_{\alpha''} \in I \setminus I' \ni (x,y) \notin T_{\alpha''}$ (since $\bigcap I = \Delta$). Now, if $T_{\alpha'''}$ is the least member of I with this property, consider $B_{\alpha'''} = \{\beta : \beta < \alpha''' \text{ and } f(T_\beta) = 1\}$. Now $(x,y) \in \bigcap_{\beta \in B_{\alpha'''}} T_\beta \nsubseteq T_{\alpha'''}$, and so certainly, for any finite set $T_{\beta_1}, \ldots, T_{\beta_n}$, $\beta_i \in B_{\alpha'''}$, we have $T_{\beta_1} \cap \ldots \cap T_{\beta_n} \nsubseteq T_{\alpha'''}$. But then, $f(T_{\alpha'''}) = 1$, which is a contradiction, and so $\bigcap I' = \Delta$.

Now (ii). Suppose we choose a finite set $T_{\alpha'_1}, \ldots, T_{\alpha'_n}$ from I', and that $T_{\alpha'_1} \cap \ldots \cap T_{\alpha'_{n-1}} \subseteq T_{\alpha'_n}$. Now $\{T_{\alpha'_1}, \ldots, T_{\alpha'_{n-1}}\} \subseteq B_{\alpha'_n}$ and so $f(T_{\alpha'_n}) = 0$, and this contradicts $f(T_{\alpha'_n}) = 1$. So we have established both required properties.

<u>Theorem 2.67</u>. If M has property X, and is compact semisimple, then M is iseomorphic to a cartesian product of finite congruence free BATS of the same type as M, with the discrete topology on each one (the canonical map, and canonical product topology being used). (If M has property X, and is compact weakly semisimple, then, we must substitute compact Hausdorff, for finite with the discrete topology.)

<u>Proof</u>. If I is the set of, open and maximal, congruences on M, we are given that $\bigcap I = \Delta$. Then, by Lemma 2.66 above, we may choose from I a well ordered set I' with the property, that, for

any finite subset $T_{\alpha_1},\ldots,T_{\alpha_n}$ from I', we have the property $T_{\alpha_1} \cap \ldots \cap T_{\alpha_{n-1}} \nsubseteq T_{\alpha_n}$. Fixing $n = \bar{n}$, we may apply this fact again to each of the sets $\{T_{\alpha_1},\ldots,T_{\alpha_t}\}$, $t = 2,\ldots,\bar{n} - 1$. So that, denoting T_{α_i} by J_i, $i = 1,\ldots,\bar{n}$, we have that $\{J_1,\ldots,J_{\bar{n}}\}$ is a set of maximal and open congruences on M such that $J_1 \cap \ldots \cap J_s \nsubseteq J_{s+1}$, $s = 1,\ldots,n - 1$. Then, by Theorem 2.59,

$$M/(J_1 \cap \ldots \cap J_{\bar{n}}) \overset{t}{\cong} M/J_1 \times \ldots \times M/J_n, \text{ or,}$$

$$M/(T_{\alpha_1} \cap \ldots \cap T_{\alpha_{\bar{n}}}) \overset{t}{\cong} M/T_{\alpha_1} \times \ldots \times M/T_{\alpha_{\bar{n}}}.$$ Thus we have a morphism

$M \twoheadrightarrow M/T_{\alpha_1} \times \ldots \times M/T_{\alpha_{\bar{n}}}$ for each $\bar{n}$ a natural number, and each $T_{\alpha_1},\ldots,T_{\alpha_{\bar{n}}}$ in I'. So the canonical map

$M \to \prod_{T_\alpha \in I'} M/T_\alpha$ attains at least one point with any finite number of predetermined coordinates. By the nature of the product topology on the right, the image of M is dense in the product. (This is true even if the T_α are merely closed.) Moreover, as is well known, the canonical function, namely $\prod_{T_\alpha \in I'} \theta_\alpha$ (where $\theta_\alpha\colon M \twoheadrightarrow M/T_\alpha$), is continuous, because each of the separate θ_α is continuous. So, since M is compact, $\left(\prod_{T_\alpha \in I'} \theta_\alpha\right)(M)$ is closed, and, being dense in the product, must be equal to it.

Theorem 2.68. Any nontrivial, compact, totally disconnected M with property X, has a nontrivial morphic image which is iseomorphic to a cartesian product of finite congruence free BATS of the same type. It also has a morphic image which is at least as large, and possibly larger, than the one above, and which is iseomorphic to a cartesian product of compact Hausdorff congruence free BATS of the same type.

Proof. We have that $M/R(M)$, and $M/\tilde{R}(M)$, from Definition 2.60, Theorem 2.61, Theorem 2.65, and Theorem 2.57 above, are nontrivial BATS, of the same type as M, which satisfy the hypotheses (or bracketed hypotheses) of Theorem 2.67. Our theorem then follows immediately.

CHAPTER III

SPECIAL PROBLEMS

1. Periodicity and Recurrence - Answers to Some Problems Posed by A. D. Wallace

In 1966, in his paper [33], and in 1968, in his seminar on Binary Topological Algebras at the University of Florida, A.D. Wallace posed two related problems, one concerning periodicity and the other concerning recurrence. We begin by defining these terms.

Definition 3.1 (Periodicity). If $f: X \to X$ is a function on the nonempty space X, we define $f^n(x)$ inductively by $f^1(x) = f(x)$, and $f^{n+1}(x) = f(f^n(x))$. Then we define f to be periodic at x in X if, for some $n \geq 1$, $f^n(x) = x$. f is periodic on X if it is periodic at each x in X.

If $(S,\cdot)$ is a semigroup, then an element a in S is defined to be periodic if, for some $n \geq 2$, $a^n = a$. S is defined to be periodic if each element is periodic.

Note that the n involved in the definitions may vary with different elements of X, or S. For future reference we define, for an element a of S, the set $O_n(a) = \{a^n, a^{n+1}, a^{n+2}, \ldots\}$. $O(a) = O_1(a)$.

Definition 3.2 (Recurrence). If $f: X \to X$ is a map (i.e. continuous function) on the compact Hausdorff space X, then f is defined to be recurrent at x in X if, for each open set $U(x)$ about x, there is an $n \geq 1$ such that $f^n(x)$ is in $U(x)$. We define f to be recurrent on X if f is recurrent at each x in X.

If $(S,\cdot)$ is a compact mob (i.e. a compact topological semigroup), then an element a in S is defined to be recurrent if, for each open set $U(a)$ about a, there is an $n \geq 2$ such that a^n is in $U(a)$. S is defined to be recurrent if each element of S is recurrent.

For future reference we define, for an element a of S, the sets $\Gamma_n(a) = O_n(a)^*$, $\Gamma(a) = O(a)^*$, and $N(a) = \bigcap_{n=1}^{\infty} \Gamma_n(a)$.

The two problems which we answer are as follows.

Problem 3.3. Is there a mathematical system, and a theorem in that system, which produces the following two theorems as corollaries?

"Theorem 1-A. If $f: X \to X$ is a function on the nonempty space X, then the following two statements are equivalent:

(i) f is periodic on X.

(ii) For each A, a subset of X,

$f(A) = \{f(x): x \in A\} \subseteq A$ implies $f(A) = A$."

"Theorem 1-B. If $(S,\cdot)$ is a semigroup then the following two statements are equivalent:

(i) S is periodic.

(ii) For each subset A, of S,

$A^2 = \{ab : a \in A, b \in A\} \subseteq A$ implies $A^2 = A$."

Problem 3.4. Is there a mathematical system, and a theorem in that system, which produces the following two theorems as corollaries?

"Theorem 2-A. If $f: X \to X$ is a map on the compact Hausdorff space X, then the following two statements are equivalent:

(i) f is recurrent on X.

(ii) For each closed subset A of X

$f(a) \subseteq A$ implies $f(a) = A$."

"Theorem 2-B. If $(S,\cdot)$ is a compact mob, then the following three statements are equivalent:

(i) S is recurrent.

(ii) For each closed subset A of S

$A^2 \subseteq A$ implies $A^2 = A$.

(iii) S is a Clifford semigroup (i.e. a union of subgroups)."

We are able to answer the two problems posed in the affirmative. At least one other person has produced a partial solution, namely L. Janos of the University of Florida. His answer [13] is of a completely different character to the one given here however, and his solution is not a perfect fit.

In each case we use an action (defined in the introduction) with a feedback function of a special type.

Theorem 3.5. Let $S \times X \to X$ be an action (discrete topology) denoted by juxtaposition, and let $\psi: X \to S$ be a function which, for

each x in X, satisfies $\psi(O(\psi(x))x) \subseteq O(\psi(x))$. Then the following two statements are equivalent:

(i) For each x in X

$x \varepsilon \{\psi(x)x, \psi(x)^2x, \ldots\} = O(\psi(x))x.$

(ii) For each $A \subseteq X$,

$\psi(A)A = \{\psi(a)\ b : a \varepsilon A, b \varepsilon A\} \subseteq A$ implies $\psi(A) = A$.

Proof. (i) implies (ii).

We suppose $A \subseteq X$ and $\psi(A)A \subseteq A$. We note that $\psi(A)^nA \subseteq A$ for each $n \geq 1$. This is seen by induction. We have the case $n = 1$ already. Supposing $\psi(A)^nA \subseteq A$ for some n, then $\psi(A)^{n+1}A = \psi(A)(\psi(A)^nA)$, since we have an action. However, this gives $\psi(A)^{n+1}A \subseteq \psi(A)A \subseteq A$. Now if $m_1 \geq m_2 \geq 1$, then $\psi(A)^{m_1}A = \psi(A)^{m_2}A$, if $m_1 = m_2$, or $\psi(A)^{m_1}A = \psi(A)^{(m_1 - m_2)}(\psi(A)^{m_2}A)$ (by the action rule) $\subseteq \psi(A)^{m_1 - m_2}A$, if $m_1 > m_2$. Let $a \varepsilon A$. Then, by (i), $a \varepsilon O(\psi(a))a$, and so, for some n, $a = \psi(a)^na \varepsilon \psi(A)^nA$. If $n = 1$ we are done. If $n > 1$, choose $m_1 = n$, and $m_2 = n - 1$ and then $\psi(A)^nA \subseteq \psi(A)A$ and $a \varepsilon \psi(A)A$ as required.

(ii) implies (i).

Choose x in X. We note that $O(\psi(x)) \cdot O(\psi(xx)) \subseteq O(\psi(x))$. We consider $A = \{x\} \cup O(\psi(x))x$, and see that

$\psi(A) = \psi(x) \cup \psi(O(\psi(x))x) \subseteq \psi(x) \cup O(\psi(x))$ (by hypothesis).

Thus $\psi(A) \subseteq O(\psi(x))$, and

$$\begin{aligned} \text{So } \psi(A)A &\subseteq O(\psi(x))\ (\{x\} \cup O(\psi(x))x) \\ &\subseteq O(\psi(x))x \cup O(\psi(x))\psi(O(\psi(x))x) \\ &\subseteq O(\psi(x))x \cup O(\psi(x))x \text{ (by the action rule)} \\ &= O(\psi(x))x \\ &\subseteq A\ . \end{aligned}$$

By (ii), this gives $\psi(A)A = A$, and thus $x \in \psi(A)A \subseteq O(\psi(x))x$, giving $x \in O(\psi(x))x$ as required.

Now it remains to show that our action with special feedback is a generalization of the situations in Problem 3.3. In Theorem 1-A take, for the S of our Theorem 3.5, the semigroup $(\{f, f^2, f^3, \ldots\}, \circ)$ of iterates of the function f, where $(f^n)\ (x)$ is defined to be $f^n(x)$, which we have already defined. Also, $f^n \circ f^m = f^{n+m}$. It is well known that this is a semigroup which acts on X by evaluation. For our function $\psi : X \to S$ take $\psi(x) = f$, for every x in X. Then

$$\begin{aligned} \psi(O(\psi(x))x) &= \psi(O(f)x) \\ &= \{f\} \\ &= \psi(x) \\ &\subseteq O(\psi(x)). \end{aligned}$$

Now f periodic, is equivalent to saying that, for each $x \in X$, $x \in O(f)x = O(\psi(x))x$. This by our Theorem 3.5 is equivalent to saying that for each $A \subseteq X$, $\psi(A)A \subseteq A$ implies $\psi(A)A = A$, which is equivalent to saying that for each $A \subseteq X$, $f(A) \subseteq A$ implies $f(A) = A$.

In Theorem 1-B take, for the X of our Theorem 3.5, the set S of the semigroup $(S,\cdot)$. Then $(S,\cdot)$ acts on S via the semigroup multiplication. For our feedback function ψ, we take $\psi(s) = s$, the identity function on S. Then

$$\begin{aligned}\psi(O(\psi(s))s &= O(\psi(s))s \\ &= O(s)s \\ &\subseteq O(s) \\ &= O(\psi(s)).\end{aligned}$$

Now S periodic is equivalent to

$$a \in O(a)a = O(\psi(a))a$$

which, by Theorem 3.5, is equivalent to having that for each $A \subseteq S$, $\psi(A)A \subseteq A$ implies $\psi(A)A = A$, and this is equivalent to saying that for each $A \subseteq S$, $A^2 \subseteq A$ implies $A^2 = A$.

Before stating and proving Theorem 3.6, we must make some definitions.

If $S \times X \to X$ is an action between the mob S, and the compact Hausdorff space X, and if there is a continuous function $\psi : X \to S$, then

$$\tilde{\Gamma}(x) = \{x, \psi(x)x, \psi(x)^2 x, \ldots\}^*$$

$$\tilde{\Gamma}_n(x) = \{\psi(x)^n x, \psi(x)^{n+1}x, \ldots\}^*, \text{ and}$$

$$\tilde{N}(x) = \bigcap_{n=1}^{\infty} \tilde{\Gamma}_n(x) \text{ (nonempty due to compactness).}$$

<u>Theorem 3.6</u>. Let $S \times X \to X$ be a mob action on the compact Hausdorff space X, and let $\psi : X \to S$ satisfy, for each x, the condition $\psi(O(\psi(x))x) \subseteq O(\psi(x))$.

Then each of the following statements is equivalent:

(i) For each x in X,

$x \in [O(\psi(x))x]^* = \tilde{\Gamma}_1(x)$.

(ii) For each $A = A^* \subseteq X$,

$\psi(A)A \subseteq A$ implies $\psi(A)A = A$.

(iii) For each x in X,

$x \in \tilde{N}(x)$, and hence $x = \bigcup_{x \in X} \tilde{N}(x)$.

(iv) For each x in X,

$x \in \tilde{N}(x)$ and $\psi(\tilde{N}(x))\, \tilde{N}(x) = \tilde{N}(x)$.

Proof. (i) implies (ii).

Let $A = A^* \subseteq X$, and $\psi(A)A \subseteq A$. Then, as in Theorem 3.5, for each $n \geq 1$, $\psi(A)^n A \subseteq \psi(A)A$. In particular, if $a \in A$, $O(\psi(a))a \subseteq \psi(A)A$. Then $[O(\psi(a))a]^* \subseteq \psi(A)A$ since, due to the continuity of ψ, and the fact that A is a closed subset of X, we get that $\psi(A)A$ is compact, hence closed. Thus, as by (i), $a \in [O(\psi(a))a]^*$ we get $a \in \psi(A)A$, and so $\psi(A)A = A$.

(ii) implies (i).

Let $x \in X$, and $A = \tilde{\Gamma}(x)$, then

$\psi(\tilde{\Gamma}(x))\, \tilde{\Gamma}(x) = \psi(\{x,\psi(x)x,\psi(x)^2x,\ldots\}^*)\ \{x,\psi(x),\ldots\}^*$

$\qquad = \psi(\{x,\psi(x)x,\psi(x)^2x,\ldots\})^*\ \{x,\psi(x),\ldots\}^*$, due

to $\{x,\psi(x)x,\ldots\}^*$ being compact in X, and ψ being continuous. Now $\psi(\tilde{\Gamma}(x))\, \tilde{\Gamma}(x) = (\psi(\{x,\psi(x)x,\psi(x)^2x,\ldots\})\ \{x,\psi(x)x,\ldots\})^*$, due to $\psi(\{x,\psi(x)x,\psi(x)^2x,\ldots\})^*$ being compact, due to the continuity of ψ. Then $\psi(\tilde{\Gamma}(x))\, \tilde{\Gamma}(x) = [(\{\psi(x)\} \cup \psi(O(\psi(x))x))\ \{x,\psi(x)x,\ldots\}]^*$

$$\subseteq \quad [(\{\psi(x)\} \cup O(\psi(x)) \quad \{x,\psi(x)x,\ldots\}]^*$$
$$= \quad [O\psi(x)\ \{x,\psi(x)x,\ldots\}]^*$$
$$\subseteq \quad [O\psi(x)x]^*$$
$$\subseteq \quad \tilde{\Gamma}(x) \quad .$$

Since $\tilde{\Gamma}(x)$ is a closed subset of X, then by (ii),

$\psi(\tilde{\Gamma}(x))\ \tilde{\Gamma}(x) = \tilde{\Gamma}(x)$, and then

$x \in \psi(\tilde{\Gamma}(x))\ \tilde{\Gamma}(x) \subseteq [O\psi(x)x]^*$, which is condition (i).

(i) implies (iv).

Let $x \in X$. We show that $x \in \tilde{\Gamma}_n(x)$ for each $n \geq 1$. Let $U(x)$ be an open set about x. In a Hausdorff space, any sequence which has a member in every open set about a fixed point has the property that, for a given open set about the same point, the sequence has an arbitrarily high member in it. So by (i), and X being Hausdorff, there is an $m > n$, such that $\psi(x)^m x \in U(x)$, so $x \in \tilde{\Gamma}_n(x)$ for each n, and so $x \in \tilde{N}(x)$.

We now show that $\psi(\tilde{N}(x))\tilde{N}(x) \subseteq \tilde{N}(x)$. Let $y \in \tilde{N}(x)$, then $y \in \Gamma_n(x)$ for each $n \geq 1$. Then, if $U(y)$ is an open set about y, $U(y)$ contains arbitrarily high members in the sequence $\{\psi(x)^n x\}_{n=1}^{\infty}$. Let y_1 and y_2 be in $\tilde{N}(x)$. Let $V(\psi(y_1)y_2)$ be an arbitrary open set about $\psi(y_1)y_2$. Then $\exists\ V_1(\psi(y_1))$, $V_2(y_2)$ open sets about $\psi(y_1)$, and y_2, respectively, such that $V_1(\psi(y_1))V_2(y_2) \subseteq V(\psi(y_1)y_2)$. Then $V'_1(y_1)$, an open set about y_1, such that $\psi(V'_1(y_1) \subseteq V_1(\psi(y_1))$. Then $\psi(V'_1(y_1))V_2(y_2) \subseteq V(\psi(y_1)y_2)$.

Now $\exists$ n_1 and n_2, natural numbers such that $\Psi(x)^{n_1} x \in V'_1(y)$, and $\psi(x)^{n2} x \in V_2(y)$. Thus $\psi(\psi(x)^{n_1}x)\ (\psi(x)^{n_2}x) \in V(\psi(y_1)y_2)$. Now $\psi(\psi(x)^{n_1}x) = \psi(x)^{n_1'}$ and $\psi(x)^{n_1'}\ (\psi(x)^{n_2}x) = \psi(x)^{n_1' + n_2}\ x$, which gives $\psi(x)^{n_1' + n_2}\ x \in V(\psi(y_1)y_2)$.

Then, by our earlier remarks, since V was arbitrary, therefore any given open set $U(\psi(y_1)y_2)$, about $\psi(y_1)y_2$, contains arbitrarily high members of the sequence $\{\psi(x)^n x\}_{n=1}^{\infty}$. Thus $\psi(y_1)y_2$ belongs to $\tilde{\Gamma}_m(x)$ for each m, and so $\psi(y_1)y_2$ is in $\tilde{N}(x)$. Now, since we are assuming (i), we have already shown that we may assume (ii), so since $\tilde{N}(x)$ is a closed subset of X, and $\psi(\tilde{N}(x))\tilde{N}(x) \subseteq \tilde{N}(x)$, we may conclude that $\psi(\tilde{N}(x))\tilde{N}(x) = \tilde{N}(x)$.

<u>(iv) implies (iii)</u>.

Obvious.

<u>(iii) implies (i)</u>.

Let $x \in \tilde{N}(x)$. Then $x \in \tilde{\Gamma}_n(x)$ for each $n \geq 1$, and so, in particular, $x \in \tilde{\Gamma}_1(x)$ as required.

Theorems 2-A, and 2-B, follow from Theorem 3.6, by making the same comparisons as for theorems 1-A, and 1-B, in relation to Theorem 3.5. We need only mention that we give $\{f, f^2, \ldots\}$ the discrete topology to get 2-A. In that case, ψ is continuous because it is a constant function. Also, in the case of 2-B, $\tilde{N}(x)$ is well known as a group because

$$\bigcap_{n=1}^{\infty} \tilde{\Gamma}_n(x) = \bigcap_{n=1}^{\infty} [O_n(\psi(x))x]^*$$

$$= \bigcap_{n=2}^{\infty} [O_n(x)]^*$$

$$= \bigcap_{n=2}^{\infty} \Gamma_n(x)$$

$$= N(x) \quad .$$

And when S is a Clifford semigroup, each element lies in a compact subgroup of S, and so, since closed subsemigroups of compact groups are themselves groups, we have $x \varepsilon \Gamma(x) = \Gamma(x) \Gamma(x) = N(x)$.

We note that we gain a strengthening of Theorem 2-A by being able to add the following two statements to the set of equivalent statements in the theorem.

(iii) For each $x \varepsilon X$, $x \in \bigcap_{n=1}^{\infty} \{f^n(x), f^{n+1}(x), \ldots\}^*$.

(iv) For each $x \varepsilon X$, $x \in \bigcap_{n=1}^{\infty} \{f^n(x), f^{n+1}(x), \ldots\}^*$ and

$$f(\bigcap_{n=1}^{\infty} f^n(x), \{f^{n+1}(x), \ldots\}^*) = \bigcap_{n=1}^{\infty} \{f^n(x), f^{n+1}(x), \ldots\}^*.$$

We are next interested to find other actions with feedback maps satisfying our condition (*)- for each $x \varepsilon X, \psi(O(\psi(x))x) \subseteq O(\psi(x))$. In particular, where our action is of a, not necessarily closed, subsemigroup of a compact semigroup S on S itself. Moreover, it is worth noting a stronger condition than (*), namely:

(#)- for each $x,y \varepsilon X$, $\psi(\psi(x)y) = \psi(x)\psi(y)$, sometimes called a right averaging operator where they occur in the theory of operators on a Banach Algebra.

<u>Theorem 3.7</u>. Condition (#) implies condition (*).

<u>Proof</u>. By induction on n.

<u>$n = 1$</u>. $\psi(\psi(x)x) = \psi(x)\psi(x)$. Suppose that for $1 \leq n \leq n_0$ we have $\psi(\psi(x)^n x) \subseteq O(\psi(x))$.

Then $\psi(\psi(x)^{n_0+1}x) = \psi(\psi(x)(\psi(x)^{n_0}x))$ (by the action property)

$= \psi(x)\psi(\psi(x)^{n_0}x)$ (by (#))

$\subseteq \psi(x)O(\psi(x))$ (by induction supposition)

$\subseteq O(\psi(x))$, and so we have

$(\psi(x)^{n_0+1}x) \subseteq O(\psi(x))$, as required.

<u>Theorem 3.8</u>. Condition (*) does not imply condition (#).

<u>Proof</u>. Take the case in Theorem 2-A. There

$\psi(\psi(x)y) = f$, and on the other hand,

$\psi(x)\psi(y) = f^2$. However, $f \neq f^2$, unless f is a retraction.

<u>Theorem 3.9</u>. If $s \times X \to X$ is an action of the mob S, on X, a compact Hausdorff space, and if $\psi: X \to S$ is a continuous function satisfying (#)- $\psi(\psi(x)y) = \psi(x)\psi(y)$, for each pair x and y in X, then $X \times X \xrightarrow{\circ} X$, defined by $x \circ y = \psi(x)\, y$, gives a compact mob $(X,\circ)$.

<u>Proof</u>. We first examine the continuity of $\circ$ from the following diagram:

$$\begin{tikzcd} X \times X \arrow[d, "\psi \times \iota"'] \arrow[dr] & \\ S \times X \arrow[r] & X \end{tikzcd}$$

$$\circ(\,(x,y)) = \mu(\,(\psi \times \iota)\,(\,(x,y)))$$

As $\psi \times 1$ is continuous, $\circ$ is the composition of two continuous functions, hence is continuous. We next examine the associativity of $\circ$.

$$\begin{aligned}
x \circ (y \circ z) &= \psi(x)\ (\psi(y)z) \\
&= (\psi(x)\psi(y))z \quad \text{(action property)} \\
&= [\psi(\psi(x)y)]z \quad \text{(by (\#))} \\
&= [\psi(x \circ y)]z \\
&= (x \circ y) \circ z.
\end{aligned}$$

We note that ψ is a morphism from $(X,\circ)$ into $(S,\cdot)$. Supposing for the moment only, that we have an action $S \times X \to X$, and a feedback map $\psi: X \to S$, then if $S \subseteq X$, we may consider whether $\psi(\psi(x)y) = \psi^2(x)\psi(y)$, a type of "homomorphism" condition. If we do indeed have such a condition satisfied, and in addition ψ is a retraction (i.e. $\psi^2 = \psi$), then we have our condition (#). In particular if $X = T$ is a semigroup, and S is a subsemigroup of T, then we get a new semigroup on T.

We now consider some particular examples.

<u>Example 3.10</u>. Let $T = \{z \in \mathbb{C}: |z| \leq 1\}$ and
$S = \{z \in \mathbb{C}: |z| \leq 1 \text{ and } z = x + 0i,\ x \text{ any real number}\}$.

Then S acts on T as a subsemigroup of T. We define $\psi: T \to S$ by $\psi(x + i\,y) = x$. Then $\psi(\psi(x + iy)\ (x' + i\,y'))$

$$\begin{aligned}
&= \psi(x(x' + i\,y')) \\
&= \psi(x\,x' + i\,x\,y') \\
&= x\,x'
\end{aligned}$$

$$= \psi(x + i\,y)\ \psi(x' + i\,y').$$

(It is also true that ψ is a retraction.)

We now define $(x + i\,y) \circ (x' + i\,y') = x(x' + i\,y')$

$$= xx' + i\,x\,y'.$$

$0 = 0 + i\,0$ is a zero for our operation $\circ$ on T. If z^* is to be a left unit for z under $\circ$, then $z^* \circ z = z$. i.e. $x^* \circ z = z$, or equivalently, $x^*\,x = x$ and $x^*\,y = y$. All elements are left units for $z = 0$, but if $z \neq 0$, then either x or $y \neq 0$, and so $x^* = 1$. Thus the only left unit for all of T, is $z = 1 + 0\,i$. If $\tilde{x} = \tilde{x} + i\,\tilde{y}$ is to be a right unit for T, then choosing $z = x + i\,y$ such that $x \neq 0$, we obtain $z \circ \tilde{x} = x\,\tilde{x} + i\,x\,\tilde{y}$. Now in order that $z \circ \tilde{z} = z$, we must have $x\,\tilde{y} = y$. But since $\tilde{y}$ is fixed, and x, y are independently variable, it is thus not possible to have a left unit for T. However, it is clear that S is again a subsemigroup of T under the $\circ$ operation, and, in fact, is isomorphic under the identity map with the original S.

Example 3.11. Let $T = \{z \in ¢: |z| \leq 1\}$,

$S = \{z = x + i\,y: y = 0,\ 0 \leq x \leq 1\}$, and define

$\Psi: T \to S$, by $\psi(z) = |z|$. Then we get

$$\begin{aligned}\psi(\psi(z_1)z_2) &= \psi(\,|z_1|z_2)\\ &= |\,|z_1|z_2|\\ &= |z_1|\ |z_2|\\ &= \psi(z_1)\ \psi(z_2).\ \text{ Also,}\end{aligned}$$

$z_1 \circ z_2 = |z_1|z_2.$

One again, 0 is a zero for $\circ$ on T. This time, all elements z, such that $|z| = 1$, are left identities for T, and so there are no right identities for T. S is a subsemigroup of T under $\circ$, and is isomorphic via the identity map with the usual S.

Example 3.12. Let $T = \{z \in ¢ : |z| \leq 1\}$ and let $S = T$. Also let $\psi(z) = -z$.

$$
\begin{aligned}
\text{Then } \psi(\psi(z_1)z_2) &= \psi(-z_1z_2) \\
&= -(-z_1z_2) \\
&= z_1z_2 \\
&= (-z_1)(-z_2) \\
&= \psi(z_1)\psi(z_2), \text{ and } z_1 \circ z_2 = -z_1z_2.
\end{aligned}
$$

$(T,\circ)$ has an identity, namely $-1 + 0\,i$, and $(T,\circ)$ has a zero, namely $0 + 0\,i$. In this case $\psi(z_1) = \psi(z_2)$, if, and only if, $z_1 = z_2$, and also, given $z \in S$, $\psi(-z) = z$, so that ψ is an iseomorphism.

Example 3.13. Let T be the set of 2 x 2 real matrices of the form $\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$, under matrix multiplication. Also, let S be the set of the form $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. Clearly S is a subsemigroup of T, and we let $\psi: T \to S$, by $\psi \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$.

Now since T is homeomorphic to $\{(x,y) : 0 \leq x + y \leq 1, 0 \leq x, 0 \leq y\}$ by $(x,y) \mapsto \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$, we get that ψ is the projection onto the x axis, and so is continuous. Moreover, if $P_1 = \begin{pmatrix} x_1 & 0 \\ y_1 & 1 \end{pmatrix}$,

and $$P_2 = \begin{pmatrix} x_2 & 0 \\ y_2 & 1 \end{pmatrix} ,$$

then $$\psi(\psi(P_1)\ P_2) = \psi \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 & 0 \\ y_2 & 1 \end{pmatrix}$$

$$= \psi \begin{pmatrix} x_1x_2 & 0 \\ y_2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \psi(P_1)\ \psi(P_2) \ ,$$

which is (#). So we may define $P_1 \circ P_2 = \psi(P_1)P_2$

$$= \begin{pmatrix} x_1x_2 & 0 \\ y_2 & 1 \end{pmatrix} .$$

However, in this case, we merely get an obvious closed subsemigroup of, $[0,1] \times [0,1]$ taken with the product multiplication from a real thread in one factor, and right trivial multiplication on the other.

The above examples all have $S \subseteq T \times X$.

Theorem 3.14. If $S \times X \to X$ is a compact action, and if $x_0 \in X$ has the property $S\ x_0 = X$, and for each s and t is S, $s\ x_0 = t\ x_0$ implies $s = t$, then $\theta\colon S \to X$ via $\theta(s) = S\ x_0$, is a homeomorphism. If we define $\psi(x) = \theta^{-1}(x)$, then ψ satisfies $\psi(\psi(x)y) = \psi(x)\psi(y)$, so that $x \circ y = \psi(x)y$ defines a mob on X.

Proof. The homeomorphism is obvious.

$$\begin{aligned}\theta(\psi(\psi(x)y)) &= \psi(x)y \\ &= \psi(x)\theta(\psi(y)) \\ &= \psi(x)(\psi(y)x_0) \\ &= (\psi(x)\psi(y))x_0 \\ &= \theta(\psi(x)\psi(y)) .\end{aligned}$$

Thus $\psi(\psi(x)y) = \psi(x)\psi(y)$.

Our semigroup $x \circ y = \psi(x)y$ will, in general, be quite different from the obvious one induced in X by the homeomorphism.

Theorem 3.15. Choose $x_0 \in S$. Then defining $\psi(x) = x\, x_0$, we get that $\psi(\psi(x)y) = \psi(x)\psi(y)$

Proof.
$$\begin{aligned}\psi(\psi(x)y) &= (\psi(x)y)x_0 \\ &= ((x\, x_0)y)\, x_0 \\ &= (x\, x_0)\,(y\, x_0) \\ &= \psi(x)\psi(y) .\end{aligned}$$

Theorem 3.16. If $(R, +, \cdot)$ is a semiring and we define $x \circ y = x\, x_0\, y$, for some fixed $x_0 \in R$, then $(R, +, \cdot)$ is also a semiring.

Theorem 3.17. More generally, if $(R, +, \cdot)$ is a semiring, and ψ is any additive morphism with the property $\psi(\psi(r)s) = \psi(r)\psi(s)$, then $(R, +, \cdot)$ is a semiring.

2. Concerning Idempotents and the "Boundary" of a Mob

During 1968, A.D. Wallace posed the following special case of a more general type of problem.

Problem 3.18. In some sense, if the sin (1/x) curve is taken with an endpoint, and limit line, then the union of the endpoint and limit line form a sort of "Boundary." Could this "Boundary " be excactly the set of idempotents for a mob on the space?

The answer is "No." In fact, we can say a little more.

Theorem 3.19. Let $x' > o$ be a fixed real number. Then the space $X = \{(x,y): y = \sin(1/x)\ 0 < x \leq x'\}$

$\cup \{(0,y): -1 \leq y \leq +1\}$, with the usual topology, does not admit a mob such that for the set of idempotents E :

$$q \in E,\ E \cap C \neq \phi \quad \text{and} \quad E \subsetneq C \cup \{q\}.$$

(Here $C = \{(0,y): -1 \leq y \leq 1\}$, and $q = (x', \sin(1/x'))$.)

Proof. By way of contradiction, suppose that it is possible. Then S is a continuum, and C has the property that any subcontinuum of X which meets C and is not contained in it, must include C. Let K be the minimal ideal of S. We know that $K \cap E$ is a subcontinuum of S. Now since $K \cap E$ is connected, it may not have q as an element as well as meet C. Suppose $K \cap C \neq \phi$. Then, if K has a point of X other than C, it must, by the properties of C and the space, contain

all of C and all points "to the left" of that point outside C. But then K has cut points, and so has either right or left trivial multiplication. But this gives nonallowable idempotents, hence a contradiction. If $q \varepsilon K$, then by a cut point argument again $K = \{q\}$. We have so far that either (i) $K \subseteq C$, or (ii) $K = \{q\}$. We next show that both of these conditions imply that S has an identity.

Case (i). Let $k \varepsilon K$. Then $k\, q = k' \varepsilon K \subseteq C$, and also $q\, q = q$, so Sq is a continuum meeting C and q, and so $S\, q = S$. Similarly, $q\, S = S$, and so $S = q\, S\, q$, and $q^2 = q$, so q is an identity for S.

Case (ii). Let $e^2 = e \varepsilon C$. Then $e\, e = e$, and $e\, q = q$, so $e\, S = S$. Similarly, $S\, e = S$, and so $S = e\, S\, e$, giving that e is an identity for S.

However, by a result of Day and Wallace [6], our space does not admit a clan, and so we have reached a contradiction and our proof is finished.

CHAPTER IV

COMPACT CONNECTED ORDERED GROUPOIDS--THE EFFECT OF VARYING DEGREES OF POWER ASSOCIATIVITY

1. Preliminaries

Ordered groupoids with some subset of their elements assumed power associative have been considered by--among many others--J. Aczel [0], K. H. Hofmann [11], L. Fuchs [7], and R. J. Warne [37]. P. S. Mostert [18] has also considered such groupoids, but a small paper which he devoted to an attempt to generalize work of Warne appears to have only a small circulation in mimeographed form.

Our work is particularly concerned with the compact connected case and began as an attempt to use methods similar to those in Warne's paper to find conditions under which such a groupoid would be iseomorphic to [0, 1/2] with the usual multiplication. As it turned out not only were we able to achieve partial success in this direction but we were able to generalize Warne's result (which is the first theorem stated below) and moreover to prove that in some sense it is a proper generalization.

While engaged in this we found an apparent error in the proof of P. S. Mostert's [18] suggested generalization of Warne's result and moreover were able to prove the existence of the error by displaying a counter example. However, it must be said that nonetheless the bulk of Professor Mostert's article did contain significant generalizations in certain directions (the mistake fortunately appeared early in the

proof), and by making use of the correct part of his proof we were able to gain a still further step in the direction that Mostert was taking.

Thanks are due to both Professors K. H. Hofmann and P. S. Mostert for the many helpful suggestions that they made for further development of our results after they had kindly consented to read a prepublication copy of a paper on part of this material. In fact, although the point of view was a little different, we had already found a number of the results outlined by Professor Hofmann before receiving his comments and we have indicated below those places where a result of his differs significantly and in a way which we had not envisaged or were unable to prove.

We begin with some definitions. An ordered groupoid is the data of an ordered set together with the order topology, in which the multiplication is continuous. An element of a groupoid is power associative if it is contained in an associative subgroupoid. If each element of the groupoid is power associative then the groupoid is called power associative. Thus we have ordered power associative groupoids (OPAGS). A sect is an OPAG which is compact and connected (of course in the order topology) which has a zero at one end ($0x = x0 = 0$, whatever x) and an idempotent, e, at the other end. Without loss of generality 0 is the least element. (Familiarity with the structure of ordered spaces is assumed, for example, R. L. Wilder [38] or Hocking and Young [10].) In all that follows when we say that an OPAG is cancellative we mean that with the exception of the least element, every element is assumed right and left cancellable. (Note: In general we do not say that the

least element is <u>not</u> cancellable, we merely do not assume that it is cancellable. Although, if it is to be a zero as is most often the case then it clearly will not be cancellable.)

For reference later we list below two well known sects which are, however, quite special in that in each case the multiplication is associative and commutative. The first is the real interval [0, 1] with ordinary real multiplication which is often termed <u>the real thread</u>, and which is a cancellative sect. The second example, often called the <u>nil thread</u> is the real interval [1/2,1] with the multiplication $x \circ y = \max\{1/2, xy\}$ where xy is ordinary real multiplication. Two groupoids are iseomorphic if there is a homeomorphism of one onto the other which is as well an algebraic homomorphism. Of course we mean our definition to exclude the trivial one element groupoid.

2. Main Development

We begin by stating Warne's result [37] from which our investigation evolved.

"Theorem. Let G be a g-thread such that G satisfies the cancellation law and the power associative elements of G form a dense subset. Then there exists a function f from G onto the unit interval [0, 1] of real numbers under the usual topology and the usual multiplication which is an isomorphism as well as an order preserving homeomorphism."

In our terminology his theorem says (since power associative elements dense means that all elements are power associative) that a cancellative sect in which the greatest element is not only an idempotent but also a unit, is iseomorphic to the real thread.

At this point we give an example to show that it is not sufficient to have just a cancellative sect, even if the additional assumptions of mediality and commutativity are made. Our example consists entirely of idempotents and is not even a semigroup.

Example 4.1. Let T be the real interval [0, 1] with the operation $x \circ y = 2xy/(x + y)$, not both x and y zero

$= 0$, otherwise.

We now check our claims.

(a) T is an algebraic groupoid on [0, 1]. This follows because, if $0 \leq x, y \leq 1$, then $0 \leq xy \leq x$ and $0 \leq xy \leq y$, so that $0 \leq 2xy \leq x + y$. Then, if neither x nor y is zero, we get $0 \leq 2xy/(x + y) \leq 1$. The other cases are trivial. (I owe to Professor Wallace the simplicity of this demonstration since at first I had a much more complicated proof.)

(b) T is a topological groupoid. It is clear that the operation is continuous at all points of $(0,1] \times (0,1]$. By writing $x \circ y = 2/((1/x) + (1/y))$ we see that by choosing x and y sufficiently small, but not 0, then we can make $x \circ y$ as close to zero as we wish.

Also for $x' \neq 0$, then we can see from $x \circ y = 2y/(1 + (y/x))$, that by choosing x sufficiently close to x', and choosing y sufficiently small, we can get $x \circ y$ as close to zero as we wish.

A similar thing may be said for $y' \neq 0$, so our operation is continuous on the set $([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$, and so on all of $I \times I$.

(c) That T is commutative is obvious.

(d) T is idempotent, hence power associative. If $x = 0$, then $x \circ x = 0$ by definition, and alternatively, if $x \neq 0$ we have $x \circ x = 2x^2 / 2x = x$.

(e) It is clear that 1 is not a unit by Warne's result but it is easily checked anyway.

$$\begin{aligned}(1/2) \circ 1 &= 2 \,.\, (1/2) \,.\, 1/((1/2) + 1 \\ &= 1/(3/2) \\ &= 2/3 \\ &\neq 1/2 \;.\end{aligned}$$

In fact, if $0 < x < 1$ then $x < 1 \circ x$, which is worth noting for later.

(d) Each nonzero element cancels. Suppose that

$$x \circ y = x \circ z, \qquad x \neq 0.$$

<u>Case 1</u>. $x \circ y = x \circ z = 0$, then $y = 0$ and $z = 0$.

Case 2. $x \circ y = x \circ z \neq 0$.

Then neither y nor z is zero and since

$$2xy \,/\, (x+y) = 2xz \,/\, (x+z), \quad \text{this implies}$$

$$y\,(x+z) = z(x+y) \quad ,$$

which then implies $xy + zy = xz + zy$

$$xy = xz$$

$$y = z \quad .$$

(e) Mediality $((a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d))$. This is trivially satisfied if any of a, b, c or d is zero. Otherwise,

$(a \circ b) \circ (c \circ d)$

$$= \frac{2(2ab/(a+b))\ (2cd/(c+d)}{(2ab/(a+b)) + (2cd/(c+d))}$$

$$= \frac{8\ ab\ cd/(a+b)(c+d)}{2\ ab\ (c+d) + 2cd\ (a+b)} \times \frac{(a+b)(c+d)}{1}$$

$$= \frac{4\ abcd}{abc + abd + cda + cdb} \quad ,$$

which remains unaltered if we replace b with c and c with b.

(f) Finally we show that our example is not associative.

$(1/2) \circ (1 \circ 1) = (1/2) \circ 1$

$$= 2/3 \quad .$$

$$((1/2) \circ 1) \circ 1 = (2/3) \circ 1$$
$$= 2.(2/3).1/((2/3) + 1)$$
$$= (4/3) \ / \ (5/3)$$
$$= 4/5 \ .$$

We now lead up to the main results by a series of lemmas and sub-lemmas.

Lemma 4.2. If T is a cancellative compact connected ordered groupoid with least element zero then

(a) $(x < y$ and $c \neq 0)$ implies $(cx < cy$ and $xc < yc)$

(b) $(x < y$ and $0 \neq w < v)$ implies $(xw < yv$ and $wx < vy)$

(b') $(x \leq y$ and $w \leq v)$ implies $(xw \leq yv)$

Proof. (a) If $x < y$ and $c \neq 0$ then supposing $cy \leq cx$, it follows that as $c[0,x]$ is an interval containing 0 and cx there must be $0 \leq q \leq x$ such that $cy = cy$, which by cancellation leads to the contradiction that $q = y$, and so we conclude that $cx < cy$. Similarly $xc < yc$, and (b) and (b') follow immediately from (a).

Lemma 4.3. If T is a cancellative compact connected ordered groupoid with least element zero, then if x is a power associative element in T and $x^2 < x$, we obtain that $x^n \to 0$ monotonically with n, and for each n, $x^n > 0$.

Proof. By Lemma 4.2 and induction, $0 < x^n < x^{n-1} < \ldots < x^2 < x$ for each n, so that $\Gamma(x) = \{x, x^2 \ldots\}^*$ contains f such that $f < x^n$ for each n, and $f = fx = ff$. Our cancellation hypothesis then requires that $f = 0$.

Lemma 4.4. If T is a sect, then for each x in T we may obtain an algebraic homomorphism from the set of strictly positive dyadic rationals, under addition, into T such that the image of 1 is x.

(We note that Professor Hofmann in his correspondence stated that it was obvious that a homomorphism could be obtained from the positive rationals into the sect with the image of 1 being x, but he did not give any proof. He did however refer before that to his paper [11], but there he had hypotheses almost equivalent to cancellation, under which the proof is indeed trivial. In any event there may be a shorter proof than the one which we are about to give but it seems doubtful, and we cannot find any place where Hofmann has proved it.)

Proof. We go by several steps.

Sub-lemma 4.4.1. If T is a sect then n-th roots exist for each x in T. Moreover for each x in T there is a least n-th root which we denote by $x^{1/n}$, and if $x < y$, then for each n, $x^{1/n} < y^{1/n}$.

Proof. That an n-th root exists for each element in T is clear because the function which raises each element to the n-th power is

continuous, thus the image of T is connected, contains 0 and e, and so must be onto T. We are unable to see that this map should be one to one (i.e., that n-th roots should be unique) unless additional assumptions are made, and the absence of uniqueness is what makes Lemma 4.4 lengthy to prove.

We continue by noting that the inverse image of a point under the above function will be a closed subset of T and hence have a least member. We denote the least n-th root of an element x in T by $x^{1/n}$. Now we suppose that $x < y$, in which case certainly $x^{1/n} \neq y^{1/n}$. If $y^{1/n} < x^{1/n}$, then there is a $0 \leq q \leq y^{1/n}$ such that $q^n = x$, by the connectivity properties of T and continuous functions. However, this would contradict the minimality of $x^{1/n}$, hence $x^{1/n} < y^{1/n}$.

<u>Sub-lemma 4.4.2</u>. $(x^{1/2^m})^{1/2^n} = x^{1/2^{m+n}}$.

<u>Proof</u>. We show that the right side is less or equal to the left and then vice versa. $(x^{1/2^m})^{1/2^n}$ is certainly a 2^{m+n}-th root of x so is greater or equal to $x^{1/2^{m+n}}$. Now $((x^{1/2^{m+n}})^{2^n})^{2^m} = x$, and so $(x^{1/2^{m+n}})^{2^n}$ is a 2^m-th root of x, which gives $x^{1/2^m} \leq (x^{1/2^{m+n}})^{2^n}$. Now $(x^{1/2^m})^{1/2^n} \leq ((x^{1/2^{m+n}})^{2^n})^{1/2^n}$ by Sub-lemma 4.4.1 and also $((x^{1/2^{m+n}})^{2^n})^{1/2^n} \leq x^{1/2^{m+n}}$ by our definition of roots. Thus $(x^{1/2^m})^{1/2^n} \leq x^{1/2^{m+n}}$ and our argument is complete.

<u>Remark</u>. A dyadic rational r may be written in the form $n/2^m$, where 2 does not divide n. However, r may also be written as $n2^q/2^{m+q}$ for any $q \geq 1$. This leads us to prove

Sub-lemma 4.4.3. $(x^{1/2^m})^n = (x^{1/2^{m+q}})^{n2^q}$.

Proof. The right side is equal to

$$((x^{1/2^{m+q}})^{2^q})^n = (((x^{1/2^m})^{1/2^q})^{2^q})^n \quad \text{(by Sub-lemma 4.4.2)}$$
$$= (x^{1/2^m})^n \ .$$

We may now make the following

Definition 4.5. If r is a dyadic rational greater than zero expressed in the form $r = n/2^m$, we define $x^r = (x^{1/2^m})^n$. (Of course $x^1 = x$.)

Thus we have a function f from the positive dyadic rationals into T defined by $f(r) = x^r$, such that $f(1) = x$. We complete our proof of Lemma 4.4 by showing that our function f defined above is a homomorphism. In other words that

$$(x^{1/2^m})^n \ (x^{1/2^k})^q = (x^{1/2^{m+k}})^{n2^k + 2^m q} \ .$$

Now $(x^{1/2^m})^n \ (x^{1/2^k})^q$

$$= (((x^{1/2^m})^{1/2^k})^{2^k})^n \ \ (((x^{1/2^k})^{1/2^m})^{2^m})^q$$
$$= (x^{1/2^{m+k}})^{n2^k} \ \ (x^{1/2^{m+k}})^{2^m q} \quad \text{(by Sub-lemma 4.4.2)}$$
$$= (x^{1/2^{m+k}})^{m2^k + 2^m q} \quad \text{(by power associativity)},$$

and so we are done.

Lemma 4.6. If T is a cancellative sect with a neighborhood V of e free of other idempotents, then any element x other than e in a half open interval $V' \subseteq V$, such that e is at the closed end of V', has the property $x^2 < x$.

Proof. Certainly $x^2 \neq x$, so we consider that possibility $x^2 > x$. Then however, using Lemma 4.2(b) and induction we have that $x < x^2 < x^3 < \ldots < x^n < \ldots < e$ a sequence monotonically increasing with n. Now by Lemma 4.2(b) $xe < ee = e$ and $x^2 < xe$, so by Lemma 4.2(b) and induction, we get that $x^n < xe < e$ for each n. Thus $\Gamma(x) = \{x, x^2, \ldots\}^*$ contains an idempotent f such that $x < f \leq xe < e$, which is a contradiction, and so we conclude that $x^2 < x$.

Lemma 4.7. If T is a cancellative compact connected OPAG with least element a zero, 0, and an element x in T such that $x^n \to 0$ monotonically with n, and $x^n \neq 0$ for each n, then every nonzero element y of T which is less than x has the same properties. Moreover $xy \leq \min\{x,y\}$ is satisfied in T if each element, except possibly the greatest, has the monotone property just described above.

Proof. By Lemma 4.2(b) and induction, $y^n < x^n$ for each n and thus $y^n \to 0$, because $x^n \to 0$. If $y^2 \geq y$, then by Lemma 4.2(b) and induction, $y \leq y^2 \leq \ldots \leq y^n \leq \ldots \leq e$, which contradicts $y^n \to 0$ and so we conclude that $y^2 < y$. Then by Lemma 4.3, y has the required properties.

Now clearly $x0 = 0 = 0x \leq \min\{0, x\}$ for each x in T. Choosing $0 < x < y$ where y is not the greatest element, then by Lemma 4.2(b), $xy < y^2 < y$, so by induction and Lemma 4.2(b), for each n we have $((\ldots((xy)\ y)\ldots)y)\ y < y^n$ (where the factor y appears n times). Supposing that $x < xy$, then $x < xy < (xy)\ y$ by Lemma 4.2(b), and then by induction and Lemma 4.2(b), we obtain for each n that $0 < x < ((\ldots((xy)\ y)\ldots)y)\ y < y^n$, which contradicts $y^n \to 0$, so we conclude that $xy \leq x$ and thus $xy \leq \min\{x,y\}$. By continuity it follows that if c is the greatest element then $xc \leq x$ and $cx \leq x$ and of course $c^2 \leq c$, so in all cases for x and y in T we have $xy \leq \min\{x,y\}$.

We remark at this point that we now have more than enough machinery to give a fairly short proof of Theorem 4.12 below. The interested reader can consult a paper by the author [24], which has been submitted for publication in the Proceedings of the American Mathematical Society (a mimeographed copy may be obtained from the author). We have however, developed machinery to handle two problems at once; one, Theorem 4.11 below, a considerable generalization on Theorem 4.12, and the other, our Theorem 4.8 about [0, 1/2]. In fact except in the method of establishing a basic homomorphism from the dyadic rationals, the proof of the theorem relating to [0, 1/2] is very similar to the one we omit in favor of greater generality. Thus we give Theorem 4.8 and its proof next, since it was stated but not proved in the paper.

<u>Theorem 4.8</u>. If T is a cancellative compact connected OPAG with a zero for the least element and a nonidempotent for the greatest

element, and if as well T is medial, then T is iseomorphic to [0, 1/2] under ordinary real multiplication.

Proof. If $x \neq 0$, then by Lemma 4.3 and Lemma 4.7, $x^n \to 0$ monotonically, and for each n, $x^n \neq 0$. Also by Lemma 4.7 we have that $xy \leq \min \{x,y\}$ is satisfied in T.

We must next construct a homomorphism from the dyadic rationals greater than or equal to one, taken with addition, into T. To do this we appear to need the conditions of cancellativity and mediality. Cancellation gives us that n-th roots, when they exist, are unique. Since not every element has a square root, we are forced to adopt the form $(x^n)^{1/2^m}$ rather than $(x^{1/2^m})^n$, and because of this a development such as in Lemma 4.4 seems to be precluded. Denoting then, the n-th root of x by $x^{1/n}$, when it exists, we note that it follows that if $n/2^m = q/2^k \geq 1$, then $(x^n)^{1/2^m}$ is defined, and $(x^q)^{1/2^k}$ is also defined and is equal to $(x^n)^{1/2^m}$.

To prove this we note first that $(x^n)^{1/2^m}$ exists for each m and n with $n \geq 2^m$. We assume that $x \neq 0$. As already noted above $x^n \leq x^{2^m}$, and so, by the connectivity properties and the fact that taking powers is a continuous function, there must be a $0 \leq t \leq x$ such that $t^{2^m} = x^n$ and so $t = (x^n)^{1/2^m}$. We suppose now that $(x^n)^{1/2^m} < (x^q)^{1/2^k}$ and $2^m \geq 2^k$. Then as noted earlier we have generally for c and d in T with $0 < c < d$, that $0 < c^n < d^n$ for each n, so in particular $x^n < ((x^q)^{1/2^k})^{2^m} = (x^q)^{2^t}$ where $t = m - k$. So we obtain that $x^n < x^{2^t q}$, and since $x \neq 0$ this means, by our remark at the beginning

of this proof, that $n > 2^t q$, but this contradicts our assumption that $n/2^m = q/2^k$. The other cases are handled similarly.

Thus we may now make:

Definition 4.9. If r is a dyadic rational greater than one expressed in any way as $n/2^m$, then we define $x^r = (x^n)^{1/2^m}$.

Clearly, we now have a function f from the dyadic rationals greater than or equal to one into T, by $f(r) = x^r$. To show that this is a homomorphism we need that

$$(x^n)^{1/2^m} (x^q)^{1/2^k} = (x^{n2^k + q2^m})^{1/2^{m+k}} .$$

We now prove a lemma which requires mediality.

Lemma 4.10. For each pair x and y in T $(x\ y)^{1/2^m}$ exists and is equal to $x^{1/2^m}\ y^{1/2^m}$, if this last expression makes sense.

Proof. We use induction on m, beginning with $m = 1$.

$$\begin{aligned} &(x^{1/2}\ y^{1/2})\ (x^{1/2}\ y^{1/2}) \\ &= (x^{1/2}\ x^{1/2})\ (y^{1/2}\ y^{1/2}) \quad \text{(by mediality)} \\ &= x\ y. \end{aligned}$$

Thus $x^{1/2}\ y^{1/2} = (x\ y)^{1/2}$. Now, if the theorem is true for $m = k$ where $1 \le k$, we show that it is true for $m = k + 1$.

$$x^{1/2^{k+1}}\ y^{1/2^{k+1}} = (x^{1/2^k})^{1/2}\ (y^{1/2^k})^{1/2} \quad \text{(by the uniqueness of roots).}$$

$$= (x^{1/2^k}\ y^{1/2^k})^{1/2}$$

(By the case $m = 1$ which was true for arbitrary x, y in T, hence for $x^{1/2^k}$ and $y^{1/2^k}$.)

$$= ((x\ y)^{1/2^k})^{1/2} \quad \text{(by the induction hypothesis)}$$

$$= (x\ y)^{1/2^{k+1}} \quad \text{(by the uniqueness of roots).}$$

$$\text{Now } (x^n)^{1/2^m}\ (x^q)^{1/2^k} = ((x^{n2^k})^{1/2^k})^{1/2^m}\ ((x^{2^m q})^{1/2^m})^{1/2^k}$$

$$= (x^{n2^k})^{1/2^{k+m}}\ (x^{2^m q})^{1/2^{m+k}}$$

$$\text{(by the uniqueness of roots)}$$

$$= (x^{n2^k} x^{2^m q})^{1/2^{m+k}} \quad \text{(by Lemma 4.10)}$$

$$= (x^{n2^k + 2^m q})^{1/2^{m+k}} .$$

At this point we remark that what we have so far is true for any nonzero x in T, but we choose to take $x = b$ the greatest element. We let $B = \{b^r:\ r \text{ is a dyadic rational greater or equal to } 1\}$.

Our last remarks have shown that B is a commutative semigroup, and also the last part of Lemma 4.7 combined with the remarks above, gives that $b^{r+s} = b^r\ b^s \leq b^r$, so that b^r decreases weakly as r increases. We next show that $B^* = T$, and then the rest follows immediately by standard arguments which may be found for example, in Aczel's paper [0].

We already have that 0 is a limit point of B. If we suppose that there is an $0 < x \leq b$ which is not a limit point of B, then there

exists $0 < x_1 < x_2 \leq b$ such that x_1 and x_2 are in B^*, but $(x_1, x_2) \cap B = \phi$. Then by Lemma 4.2, by the properties of connected sets on ordered spaces, consideration of the squaring function which is continuous, and by the definition of B, it follows that x_1^2, x_2^2 belongs to B^*, $x_1^2 < x_2^2$ and $(x_1^2, x_2^2) \cap B = \phi$. However $x_1^2 < x_1 \cdot x_2 < x_2^2$, again by Lemma 4.2, so that since B^* is a subsemigroup of T we have a contradiction. Hence x is a limit point of B.

We now give our result concerning the real thread.

Theorem 4.11. If T is a sect in which the following strong monotone property is satisfied

(*)– $0 < x < e$ implies $0 <\ldots< x^n <\ldots< x^2 < x$,

then T is iseomorphic to the real thread.

Before giving our proof we note that the main part, (iii) implies (i), of the following theorem which occurred in our paper [24], comes as a corollary to Theorem 4.11 by using Lemma 4.6 and Lemma 4.3.

Theorem 4.12. If T is a cancellative sect then the following statements are equivalent.

(i) T is iseomorphic to the real thread.

(ii) For each x in T, $xe \leq x$ (or, for each x in T $ex \leq x$).

(iii) There is a neighborhood of e free of other idempotents.

It is also interesting to note statement (ii) above, which actually can be weakened to the given inequalities holding in some neighborhood of e, because this is precisely the thing which goes wrong so badly in Example 4.1.

We begin the proof of Theorem 4.11 with:

Lemma 4.13. If T is a sect satisfying (*)— above then using the symbolism set up and justified in Lemma 4.4, if $0 < x < e$, then $x^{1/2^m} \to e$ monotonically with m, and $x^{1/2^m} < e$ for each m. Also, 0 is a limit point of the set $\{x^n\}_{n=1}^{\infty}$.

Proof. The last statement follows from the fact that $\Gamma(x)$ must contain an idempotent. Considering $x^{1/2^m}$ and $x^{1/2^{m+1}}$, by Sub-lemma 4.4.2, $x^{1/2^{m+1}} = (x^{1/2^m})^{1/2}$, and so $(x^{1/2^{m+1}})^2 = x^{1/2^m}$. Then, by the condition (*) which we assume, $(x^{1/2^{m+1}})^2 < x^{1/2^{m+1}}$, and so $x^{1/2^m} < x^{1/2^{m+1}}$. By induction then, there must be a $y \leq e$ such that $x^{1/2^m} \to$ as a limit point. Supposing that $y < e$, then since $y^{2^m} \to 0$ monotonically with m, it follows that there is a neighborhood of 0, $[0,t)$, $t < x$, and an n, such that if $m \geq n$, then y^{2^m} is in $[0,t)$. Then surely there is an open set about y, say (y_1,y_2), with $x < y_1$ and such that for each s in (y_1,y_2), s^{2^n} is in $[0,t)$. However, (y_1,y_2) must trap $x^{1/2^k}$ for all k greater than some $k \geq 2$. But then, we may choose $2^k > 2^n$ and still have $x^{1/2^k}$ in (y_1,y_2). This gives $x^{1/2^k} = (x^{1/2^{k-n}})^{1/2^n}$ (by Sub-lemma 4.4.2).

So $$(x^{1/2^k})^{2^n} = ((x^{1/2^{k-n}})^{1/2^n})^{2^n}$$

$$= x^{1/2^{k-n}}$$

$> x$, which is a contradiction. Thus $x^{1/2^n} \to e$ monotonically with n and has e as a limit point.

Lemma 4.14. If $r = n/2^m$ and $s = q/2^k$ and 2 does not divide n or q, then $n/2^m < q/2^k$ implies that

$$(x^{1/2^m})^n > (x^{1/2^k})^q .$$

Proof. Case 1. $2^m = 2^k$. Clear.

Case 2. $2^m < 2^k$. Here we may write

$x^{1/2^m} = ((x^{1/2^m})^{1/2^{k-m}})^{2^{k-m}}$, then by Sub-lemma 4.4.2

$x^{1/2^m} = (x^{1/2^k})^{n2^{k-m}}$. Also clearly $n2^{k-m} < q$ so that

$(x^{1/2^k})^{n2^{k-m}} > (x^{1/2^k})^q$ and so

$$(x^{1/2^m})^n > (x^{1/2^k})^q .$$

Case 3. $2^m > 2^k$. Then we write

$$x^{1/2^k} = ((x^{1/2^k})^{1/2^{m-k}})^{2^{m-k}}$$

$$= (x^{1/2^m})^{2^{m-k}}$$

$(x^{1/2^k})^q = (x^{1/2^m})^{2^{m-k}q}$ and

$n < 2^{m-k}q$, so $(x^{1/2^m})^n > (x^{1/2^m})^{2^{m-k}q} = (x^{1/2^k})^q$.

Thus our function from the positive dyadic rationals into T is a strictly order reversing homomorphism. We let $B = \{x^r: \ r \text{ is a dyadic rational greater than zero}\}$.
B is clearly a commutative subsemigroup of T and we need only prove that $B^* = T$ then, together with the fact that our function is order reversing, it follows easily (see for example Aczel [0]), that we have the rest.

We already have from Lemma 4.13 that 0 and e are limit points of B. Supposing by way of contradiction that there exists $0 < q < e$ which is not a limit point of B, then there must be a pair $0 < y_1 < y_2 < e$ such that y_1 and y_2 are in B^* but $(y_1, y_2) \cap B = \phi$. If r' is the real number supremum of the set of dyadic rationals r for which $x^r \geq y_2$ and r'' is the real infimum of the set of dyadic rationals s such that $x^s \leq y_1$, then it must be that $r'' = r'$.

Now there are two cases to deal with.

Case 1. y_2 is a limit point of B. Then because $x^{1/2^m} \to e$ monotonically and e is a limit point of this sequence it follows that any neighborhood of e y_2 will contain points greater than y_2 and also less than y_1 which is a contradiction. Here we have used the property that x^r is weakly decreasing as r increases.

Case 2. y_2 is not a limit point of B, in which case y_2 is an isolated point of B, but then there is a point $y_3 > y_2$ such that y_3 is the infimum in T, of the set of x^r with r a dyadic rational greater

than r'. In that case we consider an arbitrary neighborhood of ey_3 and this must contain points greater than or equal to y_2 and also points less than y_1 for the same reasons as in Case 1 and so again we have a contradiction. So we have completely contradicted our assumption on q above and it follows then, that $B^* = T$, which as remarked above completes the essentials for our proof.

We remark that Professor Hofmann, after reading our paper [24], and while we were working on the above generalization and others, proved a theorem which we had conjectured but had not quite proved at the time we received his letter. His proof draws on an advanced but nevertheless well known result of Mycielski together with some techniques developed by him in his paper [11] and in his and Mostert's jointly authored book [12]. We reproduce his theorem noting that he relies on our Lemma 4.4, which was apparently obvious to him as he omitted the proof.

Theorem 4.15 [KHH]. For any element s of a sect T, s must either be an idempotent, or be an interior point of a closed interval semigroup in T which is iseomorphic to either the real thread or the nil thread.

We come next to our discussion of Professor Mostert's paper, indicating the counter example and pointing out that the most interesting part in his proof may still be used to provide a further generalization. We also had some of this pointed out to us by Professor Hofmann, but we had already completed the proof when his letter was received. However,

he was able to add something extra, and for the sake of completeness we give some of his remarks about this as well. First Mostert's.

Conjecture 4.16. If T is a compact connected ordered groupoid in which the least element is a zero, the greatest element is a unit but which contains no other idempotents, then if as well the power associative elements have the unit as a limit point it follows that T is iseomorphic to either the real thread or the nil thread.

That this is not always the case is shown by the following

Example 4.17. Take T = [0, 1], the unit real interval and define

$$\begin{aligned} x \circ y &= 2y, && \{(x,y)\colon 0 \le x \le 1/2 \ \text{ and } \ 0 \le y \le x\} \\ & && \cup\ \{(x,y)\colon 1/2 \le x \le 1 \ \text{ and } \ 0 \le y \le 1 - x\} \\ &= 1-x+y, && \{(x,y)\colon 1/2 \le x \le 1 \ \text{ and } \ 1 - x \le y \le x\} \\ &= y \circ x, && \text{otherwise.} \end{aligned}$$

We check our claims. Everything is a matter for mental arithmetic except the continuity and the question of power associative elements having 1 as a limit point. The continuity is best seen by sketching the surface over I x I, however we leave that to the reader. Each x in $\{x\colon 1/2 \le x \le 1\}$ is power associative because in each case $x \circ x = 1$. However, the manner of the induction is slightly unusual, so we provide the full argument.

We claim that for $n \geq 1$, and n even, then for any way of introducing x as a factor n times, x^n is uniquely defined as 1. If n is odd, then we claim that x^n is always x.

By induction: We begin by writing $n = 2q - t$, where $q \geq 1$ and t is in $\{0,1\}$. Now when $n = 1$, then $q = 1$ and $t = 1$, so that $x^1 = x$. We assume our hypothesis on n to be true for all $1 \leq n \leq m$. Then $m + 1 = 2q - t$ for some $q \geq 1$, and some t in $\{0,1\}$.

Our final product must be made of a pair of factors x^{n_1} and x^{n_2} respectively (well defined by hypothesis since each of n_1 and n_2 is less than or equal to m).

There are two cases:

Case 1. $m + 1$ is even, then either

1.1 both n_1 and n_2 are even, in which case we have by hypothesis that $x^{n_1} \circ x^{n_2} = 1 \circ 1 = 1$ (by definition),

or 1.2 n_1 and n_2 are both odd in which case we have by hypothesis $x^{n_1} \circ x^{n_2} = x \circ x = 1$ (by definition).

So that always if $m + 1$ is even we get that x^{m+1} is defined and equal to 1.

Case 2. $m + 1$ is odd, then either

2.1 n_1 is odd and n_2 is even, in which case we get $x^{n_1} \circ x^{n_2} = x \circ 1 = x$ (by definition),

or 2.2 n_1 is even and n_2 is odd, then $x^{n_1} \circ x^{n_2} = 1 \circ x = x$ (by definition). So always if $m + 1$ is odd we get that x^{m+1} is defined and equal to x.

Thus always x^{m+1} is defined and has the stated values and so our induction is complete.

We note that it follows as a corollary to Professor Hofmann's Theorem (Theorem 4.15 above) that if all elements of the groupoid are power associative then the conjecture is true, even if the greatest element is assumed only to be an idempotent. We had conjectured as much and were working on it but were beaten to the proof.

Our next result was obtained independently working from the outline in the correct portion of Mostert's paper, and some of the working which we used is also to be found in Hofmann's paper [11] (which we were not aware of until later), so it is omitted.

Theorem 4.18. If T is a compact connected ordered groupoid with a zero, 0, for a least element, with a weakly cancellative idempotent, e, for greatest element ($x \neq y$ implies that either $ex \neq ey$ or $xe \neq ye$), but with no other idempotents, and if the power associative elements cluster at e but their powers do not, then T is iseomorphic to either the real thread or the nil thread.

We remark that the weak cancellation property of e seems necessary in order to get that e is a unit and we do not, at this point, share Professor Mostert's belief, communicated to us recently, that it might be done away with, although this may be the case.

Apparently Professor Hofmann did not see a way either for he gave the above result in the case where e is a unit.

However, stimulated by our counter example he produced a nice statement of the case when the powers of some power associative elements can cluster at 1. We too, had arrived at similar conclusions, but had not yet completed our proof when his arrived. We omit the proof but state his result.

Theorem 4.19 [KHH]. If T is a compact connected ordered groupoid with least element a zero, 0, and greatest element a unit, 1, and with no other idempotents, then if as well the power associative elements cluster at 1, exactly one of the following cases obtains:

(i) T is iseomorphic to the real thread,

(ii) T is iseomorphic to the nil thread,

(iii) 1 is an accumulation point of square roots of 1.

We state a further result of ours before branching off into more general considerations concerning power associative groupoids.

Theorem 4.20. Let T be a cancellative compact connected OPAG (not assumed to have zero for least element) with an idempotent, e, for greatest element. If e is isolated in the set of idempotents and if there exist elements t and t' in T such that $x < y$ implies $tx < ty$ and $xt' < yt'$, then T is iseomorphic to the real thread.

Proof. We remind the reader of our definition of cancellative as meaning that every element except possibly the least is cancellative. The proof is straightforward but rather lengthy and so is omitted.

3. Additional Remarks

The theorems and examples which follow were obtained in response to various questions posed by Professor A. D. Wallace while seeking to characterize those OPAGS which are semigroups (not necessarily the real thread or the nil thread).

Theorem 4.21. A compact connected ordered semigroup T with a zero, 0, for least element and with the property that the squaring function is identically zero, must have $T^3 = \{0\}$.

Proof. Choosing any triple a, b and c from T then either $ab \leq bc$ or $ab \geq bc$. Supposing $ab \leq bc$, then there is $0 \leq q \leq b$ such that $ab = qc$. Then $(ab)c = (qc)c = q(c^2) = q0 = 0$. A similar argument applies if $bc \leq ab$. Thus, we have $T^3 = \{0\}$.

The following example shows that it will not in general be true that $T^2 = \{0\}$ under the hypotheses of Theorem 4.21, nor will it follow that T is commutative.

Example 4.22. Set $T = [0,1]$ the real interval, with the operation

$$
\begin{aligned}
x \circ y &= 0 && \{(x,y)\colon 0 \le x \le 1 \text{ and } 0 \le y \le 1/2\} \\
& && \cup \{(x,y)\colon 0 \le x \le 1/2 \text{ and } 1/2 \le y \le 1\} \\
&= (1/2)(y-(1/2)), && \{(x,y)\colon 1/2 \le x \le 1 \text{ and } \\
& && 1/2 \le y \le 1/8\ (x + (7/2))\} \\
&= (1/2)(x-y), && \{(x,y)\colon 1/2 \le x \le 1 \text{ and } \\
& && (1/8)(x + (7/2)) \le y \le x\} \\
&= y-x, && \{(x,y)\colon 1/4 \le x \le 3/4 \text{ and } \\
& && x \le y \le 2(x - (1/4))\} \\
& && \cup \{(x,y)\colon 3/4 \le x \le 1 \text{ and } x \le y \le 1\} \\
&= x - (1/2), && \{(x,y)\colon 1/2 \le x \le 3/4 \text{ and } \\
& && 2(x - (1/4)) \le y \le 1\} .
\end{aligned}
$$

We note that the continuity is best seen from a diagram over $I \times I$ and leave that to the reader. Now it may be seen that always $x \circ y < 1/2$ and whenever $p \le 1/2$ then $p \circ q = q \circ p = 0$, whatever q. Hence $(x \circ y) \circ z = 0$ and also $x \circ (y \circ x) = 0$, whatever x, y and z, so we have associativity. However, although inspection shows that $x \circ x$ is always zero, $x \circ y$ is not always zero. For example: If $x = 1$ and $y = 3/4$, then $x \circ y = 1/8$. Also in this case $y \circ x = 1/4$, which shows that T is not commutative.

The next example shows how difficult it is to characterize semigroups on $[0,1]$ amongst all the possible topological groupoids,

unless, as well as having 0 a zero, we have 1 at least an idempotent, when of course we tend to get all the way to either the real thread or the nil thread as amply demonstrated before.

The next example shows that a topological groupoid, T, on the real interval [0,1], even under the following apparently stringent conditions

(a) power associativity,

(b) commutativity,

(c) 0, a zero for T, and no other idempotents,

(d) an element x such that $x \circ x \neq 0$,

(e) $x \neq 0$ and $y \neq 0$ implies $x \circ y < \min\{x,y\}$, still need not be a semigroup!

<u>Example 4.23.</u> $T = [0,1]$ the real interval with

$$
\begin{aligned}
x \circ y &= 0, && \{(x,y)\colon 0 \le x \le 1 \text{ and } 1/4 \le y \le 1\} \\
&= 2(y - (1/4)), && \{(x,y)\colon 1/4 \le x \le 1 \text{ and} \\
& && 1/4 \le y \le (1/3)(x + (1/2))\} \\
&= (1/24)(22x-18y-1), && \{(x,y)\colon 1/4 \le x \le 1 \text{ and} \\
& && (1/3)(x + (1/2)) \le y \le x\} \\
&= y \circ x, && \text{otherwise.}
\end{aligned}
$$

We check our claims.

Once again the question of continuity is left to the reader. Clearly, we defined our groupoid to be commutative, and it is easily seen that 0 acts as a zero, that there are no other idempotents, and

that if $1/4 \leq x$, then $x \circ x = 0$. It is not so easy to see, unless a diagram is drawn, that (e) is satisfied but still we leave it to the reader.

We restrict our attention to point (a) above. First we remark that always $x \circ x \leq 1/8$, and that if $p \leq 1/4$, then $p \circ q = q \circ p = 0$, whatever q. Thus $(x \circ x) \circ y = y \circ (x \circ x) = 0$, for any y in T. From now on we denote by x^n the result of combining n factors, each one x, in some order via the circle operation. We show now that x^n is well defined as zero if $n \geq 3$. Since $x^3 = x \circ (x \circ x) = (x \circ x) \circ x = 0$, by our remarks above taking $y = x$, then we have our first step in the induction. Supposing then, that x^n is well defined as zero for $3 \leq n \leq m$, then whatever way "x^{m+1}" is written it must be the product of a pair of factors x^{m_1} and x^{m_2} where $1 \leq m_1 \leq m$ and $1 \leq m_2 \leq m$. The factors are well defined by definition if the exponents are either 1 or 2, and in case the exponents are greater or equal 3, then the factors are zero by the induction hypothesis. There are several cases.

<u>Case 1.</u> $m_1 = 1, \quad m_2 = m$

$$x \circ (x^m) = x \circ x = 0$$

(by the induction hypothesis) .

<u>Case 2.</u> $m_1 = m, \quad m_2 = 1$

$$(x^m) \circ x = 0 \circ x = 0$$

(by the induction hypothesis) .

<u>Case 3.</u> $m_1 = 2, \quad m_2 = m - 1$

$$(x \circ x) \circ (x^{m-1}) = (x \circ x) \circ y = 0$$

(by earlier remarks taking $y = x^{m-1}$).

Case 4. $m_1 = m - 1, \quad m_2 = 2$

$$(x^{m-1}) \circ (x \circ x) = y \circ (x \circ x) = 0$$

(by earlier remarks taking $y = x^{m-1}$).

Case 5. $m_1 \geq 3, \quad m_2 \geq 3$

$$(x^{m_1}) \circ (x^{m_2}) = 0 \circ 0 = 0$$

(by induction hypothesis).

Finally we show that the operation is not associative. We choose $x = 11/20$, $y = 1/2$ and $z = 1$. Then

$$\begin{aligned}(x \circ y) \circ z &= [(1/24)(22\ (11/20) - 18\ (1/2) - 1)] \circ 1 \\ &= [(1/24)((121/10) - 9 - 1)] \circ 1 \\ &= [(1/24)(12.1 - 10.0)] \circ 1 \\ &= ((1/24)(2.1)) \circ 1 \\ &= q \circ 1, \quad \text{where } q < 1/4 \quad .\end{aligned}$$

Thus $(x \circ y) \circ z = 0$, by our earlier remarks.

$$\begin{aligned}x \circ (y \circ x) &= (11/20) \circ ((1/2) \circ 1) \\ &= (11/20) \circ (1 \circ (1/2)) \\ &= (11/20) \circ 2((1/2) - (1/4)) \\ &= (11/20) \circ (1/2) \\ &= (1/24)\ (2.1) \quad \text{(from above)} \\ &> 0 \quad .\end{aligned}$$

Thus $(x \circ y) \circ z \neq x \circ (y \circ z)$.

Further to our second last example we note the following

Theorem 4.24. If T is a compact connected ordered semigroup in which for each x in T, $x^2 = c$ for some fixed c in T, then $T^2 \neq T$.

Proof. By a result of Koch and Wallace [16], if we suppose that $T^2 = T$, then since we have that the only idempotent in T is c, it follows that T is a topological group, but this is not possible since the topology of T is nonhomogeneous, and so we reach a contradiction.

We conclude with the following result which is proved by a combination of work from Wallace [32], Koch and Wallace [16], Storey [29] and others, and is of interest due to work of Clifford [3] [4] in this area.

Theorem 4.25. If S is a compact connected ordered semigroup such that $S^2 = S$, then at least one of the following is true,

(i) the endpoints are idempotents,

(ii) one of the endpoints is a one sided unit.

Proof. If the minimal ideal of S meets either end of S, then since it consists entirely of idempotents, due to either being a zero or having cutpoints, we are either done immediately if K equals S, or if not we still get our result due to the maximal ideal being dense and connected, since then the complement must be an idempotent. Consider then the case when the (connected) minimal ideal meets neither end. From now on the greatest element of S will be called b, and the least element a.

We now assume that the squaring function is onto. Before we proceed further it is best if we state part of a result of Storey. Before doing so however, we remind the reader of certain useful terminology used by Storey. We denote

$R = \{t\colon k \le t \text{ for each } k \text{ in } K\}$

$L = \{t\colon t \le k \text{ for each } k \text{ in } K\}$.

Storey's result says then in part, that if S is a compact connected ordered semigroup (which always has a minimal ideal K which is compact and connected) such that $S^2 = S$, then either $R^2 = R$ or $L^2 = L$ and K does not separate R^2, L^2, LR, or RL. We hasten to add that Storey's Theorem says a good deal more, but we have stated the part useful to us. Now, if the squaring function is onto and the minimal ideal meets neither end of S, then since neither R^2 nor L^2 is separated by K, it must be that the endpoints are the squares of elements one from L and the other from R. Since either $R^2 = R$, or $L^2 = L$, it must be that all squareroots of a are in L and all squareroots of b are in R. Letting $q^2 = a$ and denoting by k_1 the least element of K, we get that $q[q,k_1] \supseteq [a,k_1]$ and so by the Swelling Lemma of Wallace, we have $q = a$. Similarly if $s^2 = b$ and if k_2 is the greatest element of K, then $s\,[k_2,s] \supseteq [k_2,b]$ and so $s = b$.

Supposing now that the image of the squaring function meets only one end of S (without loss of generality the least end), but still, K meets neither end. Clearly, in this case the complement of a maximal ideal cannot be b alone or it would have to be an idempotent which we

have supposed above it is not. If it were the least element alone then $a^2 = a$ and by Storey's Theorem again there would be a t in R and an x in S such that either $xt = b$ or $tx = b$ (without loss of generality $xt = b$). Now if $k_2 < x$ then by the Swelling Lemma again, since $x\,[k_2,t] \supseteq [k_2,b]$ then $t = b$. However also $[k_2,x]\,t \supseteq [k_2,b]$ and so $x = b$, and we have altogether, that $b^2 = b$ which again is our result. If now instead, $x < k_1$, (still with $k_2 < t$) then again $t = b$. However, in this case we show that $ab = b$ which again is our result. The Swelling Lemma is still the tool, for a double application of it gives that $ax = x$. Then $ab = a\,(xb) = (ax)\,b = xb = b$. Now possibly still if the image of the squaring function meets only the least end, and the minimal ideal neither end, it might be that the complement of a maximal ideal is exactly both ends. However, then either $a^2 \neq a$ in which case either $ac = a$ and $ca = b$ giving $a \neq aa = (ac)a = a(ca) = ac = a$, which is a contradiction, or $ac = b$ and $ca = a$ also giving a contradiction; <u>or</u> $a^2 = a$ and then either $ab = b$ or $ba = b$ which is still our result.

Finally we have the case where the image of the squaring function meets neither end of S. Then plainly the complement of a maximal ideal cannot be just one end. Supposing then that it was the two ends. In this case, of necessity either $ab = a$ and $ba = b$ when we get $a^2 = (ab)\,a = a\,(ba) = ab = a$, which is a contradiction, or $ba = a$ and $ab = b$ which also gives a contradiction, and so we are done.

Remark. We discovered after proving this last result that Paalman- de Miranda has a slightly different proof of a similar result.

BIBLIOGRAPHY

[0] Aczel, J., "Quasigroups, nets and nomograms," Advances in Mathematics, 1(1965), 383-449.

[1] Bednarek, A. R., and Wallace, A. D., "Equivalences on machine state spaces," Math. Casopis, 17(1967), 1-7.

[2] Beidleman, J. C., and Cox, R. H., "Topological near-rings," Arch. der Math., 18(1967), 485-492.

[3] Clifford, A. H., "Connected ordered topological semigroups with idempotent endpoints I," Trans. Amer. Math. Soc., 88(1958), 80-98.

[4] ______________, "Connected ordered topological semigroups with idempotent endpoints II," Trans. Amer. Math. Soc., 91(1959), 193-208.

[5] Clifford, A. H., and Preston, G. B., "The algebraic theory of semigroups, Vol. I," Mathematical Surveys 7, Amer. Math. Soc., 1961.

[6] Day, J. M., and Wallace, A. D., "Semigroups acting on continua," J. Austral. Math. Soc., 7(1967), 327-340.

[7] Fuchs, L., Partially Ordered Algebraic Systems, Pergamon Press, Addison-Wesley, 1963, 181-188.

[8] Faucett, W. M., Koch, R. J., and Numakura, K., "Complements of maximal ideals in compact semigroups," Duke Math. J. 22(1955), 655-661.

[9] Hildebrandt, J. A., "Compact divisible abelian semigroups," Proc. Am. Math. Soc., 19(1968), 405-410.

[10] Hocking, J. G., and Young, G. S., Topology, Addison-Wesley, 1961.

[11] Hofmann, K. H., "Zur mathematischen Theorie des Messens," Rozprawy Matematiczne, 32(1963), 32 pages.

[12] Hofmann, K. H., and Mostert, P. S., Elements of Compact Semigroups. Merrill, 1966.

[13] Janos, L., "A generalization of Clifford semigroup," Arch. Math. (Basel), 21(1970), 125-127.

[14] Kelley, J. L., General Topology. Van Nostrand, 1955.

[15] Kimura, N., "The structure of idempotent semigroups (I)," Pacific J. Math., 8(1958), 257-275.

[16] Koch, R. J., and Wallace, A. D., "Maximal ideals in compact semigroups," Duke Math. J., 21(1954), 681-686.

[17] Montgomery, D., and Zippin, L., Topological Transformation Groups. Interscience Publishers, 1955.

[18] Mostert, P. S., "Comments on a paper of Warne," Mimeographed sheets, Tulane University, c. 1963.

[19] Numakura, K., "Theorems on compact totally disconnected semigroups and lattices," Proc. Amer. Math. Soc., 8(1957), 623-626.

[20] Paalman- de Miranda, A. B., "Topological semigroups," Mathematisch Centrum, 11(1964), 169 pages.

[21] Pearson, K. R., "Compact semirings which are multiplicatively groups or groups with zero," Math. Zeitschr., 106(1968), 388-394.

[22] ________________, "The three kernels of a compact semiring," J. Austral. Math. Soc., 10(1969), 299-319.

[23] Robbie, D. A., "Some recent results in topological semirings and nearrings," Mimeographed sheets, University of Florida, 1970.

[24] ________________, "Ordered power associative groupoids," Proc. Amer. Math. Soc. (submitted).

[25] Selden, J., Jr., "Theorems on topological semirings," Dissertation, University of Georgia, 1963.

[26] ________________, "A note on compact semirings," Proc. Amer. Math. Soc., 15(1964), 882-886.

[27] Sigmon, K. N., "Cancellative medial means are arithmetic," Duke Math. J., 38(1970), 439-445.

[28] Stadtlander, D. P., "Semigroup action on topological spaces," Dissertation, Pennsylvania State Univ., 1966.

[29] Storey, C. R., "The structure of threads," Pacific J. Math., 10(1960), 1429-1445.

[30] Wallace, A. D., "On the structure of topological semigroups," Bull. Amer. Math. Soc., 61(1955), 95-112.

[31] Wallace, A. D., "The Rees-Suschkewitsch structure theorem for compact simple semigroups," Proc. Nat. Acad. Soc., 42 (1956), 430-432.

[32] ______________, "Relative ideals in semigroups, I (Faucett's Theorem)," Coll. Math., 9(1962), 55-61.

[33] ______________, "Problems on periodicity--functions and semigroups," Math. Casopis, 16(1966), 209-212.

[34] ______________, "Externally induced operations," Mathematical Systems Theory 3(1969), 244-245.

[35] ______________, "Dispersal in compact semimodules," J. Austral. Math. Soc. (submitted).

[36] ______________, "Binary topological algebra notes," Mimeographed sheets, University of Florida, 1969.

[37] Warne, R. J. "Connected ordered topological groupoids with idempotent endpoints," Pub. Math. Debrecen, 8(1961), 143-146.

[38] Wilder, R. L., "Topology of manifolds," Amer. Math. Soc., Colloquium Publ., Vol. 32, 1949.

BIOGRAPHICAL SKETCH

Desmond Alexander Robbie was born September 15, 1938, at Horsham, Victoria, Australia and lived nearby at Goroke during his early years. He graduated from Stawell High School in December, 1956, obtaining First Class Honours in Physics and Chemistry on the University of Melbourne Matriculation Examinations. In March, 1960, he received the degree of Bachelor of Science from the University of Melbourne and after some further studies in Education he taught in Victoria for five years, four of them at Haileybury College. He received the post-graduate degree of Bachelor of Education in March, 1964, from the University of Melbourne as a result of part time studies. He returned later to full time studies, and in April, 1968, he received the Honours degree of Bachelor of Science from Monash University obtaining First Class Honours at the final honours examinations for double majors in Pure Mathematics. In January, 1968, he enrolled in the Graduate School of the University of Florida. He worked in the Mathematics Department there, sometimes as a graduate assistant and sometimes as a research assistant, and in the year 1969-1970 he was a University of Florida Graduate School Fellow. From enrollment to the present time he has pursued his work towards the degree of Doctor of Philosophy.

Desmond Alexander Robbie is married to the former Judith Adele Bridge, and has one daughter, Melissa Jane Robbie. He is a member of the American Mathematical Society and the Australian Mathematical Society.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Alexander D. Wallace, Chairman
Graduate Research Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Kermit N. Sigmon, Co-Chairman
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David P. Stadtlander
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Alexander R. Bednarek
Professor of Mathematics and
Chairman of Department

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Janet Ault

Janet Ault
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Charles F. Hooper, Jr.

Charles F. Hooper, Jr.
Associate Professor of Physics

This dissertation was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1970

Dean, College of Arts and Sciences

Dean, Graduate School